5.4　3 重積分

問 1　求める体積を V とおく．

(1)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1-x,\ 0 \leqq z \leqq 1-x-y$ とすると，対称性より，

$$V = 8\iiint_D dxdydz = 8\int_0^1 dx \int_0^{1-x}\int_0^{1-x-y} dz = 8\int_0^1 dx\int_0^{1-x}(1-x-y)dy$$
$$= 4\int_0^1 (1-x)^2 dx = \frac{4}{3}.$$

(2)　$D\colon x^2+y^2 \leqq 1,\ 0 \leqq z \leqq \sqrt{1-x^2-y^2}$ とし，$E\colon x^2+y^2 \leqq 1$ とおく．対称性より，

$$V = 2\iiint_D dxdydz = 2\iint_E \int_0^{\sqrt{1-x^2-y^2}} dz = 2\iint_E \sqrt{1-x^2-y^2}dxdy$$
$$= 4\pi\int_0^1 \sqrt{1-r^2}rdr = 4\pi\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}\pi.$$

(3)　$D\colon 0 \leqq x \leqq 3,\ 0 \leqq y \leqq \sqrt{9-x^2},\ 0 \leqq z \leqq \sqrt{9-x^2}$ とすると，対称性より，

$$V = 8\iiint_D dxdydz = 8\int_0^3 dx\int_0^{\sqrt{9-x^2}} dy \int_0^{\sqrt{9-x^2}} dz = 8\int_0^3 \left(9-x^2\right)dx = 8\left[9x - \frac{x^3}{3}\right]_0^3 = 144.$$

(4)　$D\colon x^2+y^2 \leqq 1,\ x \leqq z \leqq 2x$ とし，$E\colon x^2+y^2 \leqq 1,\ x \geqq 0$ とおくと，

$$V = \iiint_D dxdydz = \iint_E xdxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^1 r^2\cos\theta dr = [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^1 = \frac{2}{3}.$$

(5)　$D\colon \dfrac{x^2}{9}+\dfrac{y^2}{16} \leqq 1,\ 0 \leqq z \leqq 5\sqrt{1-\dfrac{x^2}{9}-\dfrac{y^2}{16}}$ とし，$E\colon \dfrac{x^2}{9}+\dfrac{y^2}{16} \leqq 1$ とおくと，対称性より，

$$V = 2\iiint_D dxdydz = 10\iint_E \sqrt{1-\frac{x^2}{9}-\frac{y^2}{16}}dxdy.$$

ここで，$x = 3r\cos\theta$，$y = 4r\sin\theta$ と変換して，

$$V = 120\int_0^{2\pi} d\theta\int_0^1 \sqrt{1-r^2}rdr = 240\pi\left[-\frac{1}{3}\left(1-r^2\right)^{\frac{3}{2}}\right]_0^1 = 80\pi.$$

(6)　$D\colon (x-1)^2+y^2 \leqq 1,\ y \geqq 0,\ 0 \leqq z \leqq \sqrt{4-x^2-y^2}$ とし，$E\colon (x-1)^2+y^2 \leqq 1,\ y \geqq 0$ とおくと，対称性より，

$$V = 4\iiint_D dxdydz = 4\iint_E \sqrt{4-x^2-y^2}dxdy$$

なので，極座標変換して，

$$V = 4\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} \sqrt{4-r^2}rdr = 4\int_0^{\frac{\pi}{2}}\left[-\frac{1}{3}\left(4-r^2\right)^{\frac{3}{2}}\right]_0^{2\cos\theta} d\theta$$
$$= \frac{32}{3}\int_0^{\frac{\pi}{2}}\left(1-\sin^3\theta\right)d\theta = \frac{32}{3}\left[\theta + \frac{3}{4}\cos\theta - \frac{1}{12}\cos 3\theta\right]_0^{\frac{\pi}{2}} = \frac{16}{9}(3\pi-4).$$

問 2

(1) $\displaystyle \iint_D x^2y^3z^4 dxdydz = \left(\int_0^1 x^2dx\right)\left(\int_{-3}^0 y^3dy\right)\left(\int_0^2 z^4dz\right) = \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^4}{4}\right]_{-3}^0 \left[\frac{z^5}{5}\right]_0^2 = -\frac{216}{5}$

(2) $\displaystyle \iiint_D (x+y+z)^4 dxdydz = \int_{-1}^1 dx \int_{-1}^1 dy \int_{-1}^1 (x+y+z)^4 dz$

$$= \int_{-1}^1 dx \int_{-1}^1 \left[\frac{1}{5}(x+y+z)^5\right]_{z=-1}^{z=1} dy = \frac{1}{5}\int_{-1}^1 dx \int_{-1}^1 \left\{(x+y+1)^5-(x+y-1)^5\right\} dy$$

$$= \frac{1}{5}\int_{-1}^1 \left[\frac{1}{6}\left\{(x+y+1)^6-(x+y-1)^6\right\}\right]_{y=-1}^{y=1} dx$$

$$= \frac{1}{30}\int_{-1}^1 \left\{(x+2)^6-2x^6+(x-2)^6\right\} dx = \frac{1}{30}\left[\frac{1}{7}\left\{(x+2)^7-2x^7+(x-2)^7\right\}\right]_{-1}^1 = \frac{104}{5}$$

(3) $\displaystyle \iiint_D \sqrt{x+y-z}\, dxdydz = \int_{-2}^1 dx \int_1^3 \int_{-4}^{-2} \sqrt{x+y-z}\, dz$

$$= \int_{-2}^1 dx \int_1^3 \left[-\frac{2}{3}(x+y-z)^{\frac{3}{2}}\right]_{z=-4}^{z=-2} dy$$

$$= -\frac{2}{3}\int_{-2}^1 dx \int_1^3 \left\{(x+y+2)^{\frac{3}{2}}-(x+y+4)^{\frac{3}{2}}\right\} dy$$

$$= -\frac{2}{3}\int_{-2}^1 \left[\frac{2}{5}\left\{(x+y+2)^{\frac{5}{2}}-(x+y+4)^{\frac{5}{2}}\right\}\right]_{y=1}^{y=3} dx$$

$$= -\frac{4}{15}\int_{-2}^1 \left\{2(x+5)^{\frac{5}{2}}-(x+7)^{\frac{5}{2}}-(x+3)^{\frac{5}{2}}\right\} dx$$

$$= -\frac{4}{15}\left[\frac{2}{7}\left\{2(x+5)^{\frac{7}{2}}-(x+7)^{\frac{7}{2}}-(x+3)^{\frac{7}{2}}\right\}\right]_{-2}^1$$

$$= \frac{8}{105}\left(127+1024\sqrt{2}+54\sqrt{3}-125\sqrt{5}-432\sqrt{6}\right).$$

(4) $D\colon 0 \leqq x \leqq 2,\ 0 \leqq y \leqq 2-x,\ 0 \leqq z \leqq 2-x-y$ より，

$$\iiint_D (x+y+z)^3 dxdydz = \int_0^2 dx \int_0^{2-x} dy \int_0^{2-x-y} (x+y+z)^3 dz$$

$$= \int_0^2 dx \int_0^{2-x} \left[\frac{(x+y+z)^4}{4}\right]_{z=0}^{z=2-x-y} dy = \frac{1}{4}\int_0^2 dx \int_0^{2-x} \left\{16-(x+y)^4\right\} dy$$

$$= \frac{1}{4}\int_0^2 \left\{16(2-x) - \left[\frac{(x+y)^5}{5}\right]_{y=0}^{y=2-x}\right\} dx = \frac{1}{20}\int_0^2 \left(128-80x+x^5\right) dx$$

$$= \frac{1}{20}\left[128x-40x^2+\frac{x^6}{6}\right]_0^2 = \frac{16}{3}.$$

(5) $D\colon 0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \pi-x,\ 0 \leqq z \leqq \pi-x-y$ より，

$$\iiint_D \cos(x+y+z) dxdydz = \int_0^\pi dx \int_0^{\pi-x} dy \int_0^{\pi-x-y} \cos(x+y+z) dz$$

$$= \int_0^\pi dx \int_0^{\pi-x} [\sin(x+y+z)]_{z=0}^{z=\pi-x-y} dy = -\int_0^\pi dx \int_0^{\pi-x} \sin(x+y) dy$$

$$= \int_0^\pi [\cos(x+y)]_{y=0}^{y=\pi-x} dx = \int_0^\pi (-1-\cos x)\, dx = [-x-\sin x]_0^\pi = -\pi.$$

(6)　$D\colon 0 \leqq x \leqq 1,\, 0 \leqq y \leqq 1-x,\, 0 \leqq z \leqq 1-x-y$ より，

$$\iiint_D (x^2+y^2+z^2)dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x^2+y^2+z^2)dz$$
$$= \int_0^1 dx \int_0^{1-x} \left[(x^2+y^2)z + \frac{z^3}{3}\right]_{z=0}^{z=1-x-y} dy$$
$$= \int_0^1 dx \int_0^{1-x} \left\{(1-x)x^2 - x^2y + (1-x)y^2 - y^3 + \frac{(1-x-y)^3}{3}\right\} dy$$
$$= \int_0^1 \left[(1-x)xy - \frac{x^2}{2}y^2 + \frac{1-x}{3}y^3 - \frac{y^4}{4} - \frac{(1-x-y)^4}{12}\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left\{\frac{x^2}{2} - x^3 + \frac{x^4}{2} + \frac{(1-x)^4}{6}\right\} dx = \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10} - \frac{(1-x)^5}{30}\right]_0^1 = \frac{1}{20}.$$

問 3

(1)　極座標変換より，

$$\iiint_D (x^2+y^2+z^2)^2 dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^2 r^6 \sin\theta dr$$
$$= \left(\int_0^2 r^6 dr\right)\left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\varphi\right) = 2\pi\left[\frac{r^7}{7}\right]_0^2 [-\cos\theta]_0^{\pi} = \frac{512}{7}\pi.$$

(2)　$x = \dfrac{-u+9v+3w}{2}$，$y = v$，$z = \dfrac{u-3v-w}{2}$ により，D は $E\colon -\pi \leqq u \leqq 0,\, |v| \leqq 1,\, -\dfrac{\pi}{2} \leqq w \leqq \dfrac{\pi}{2}$ に写り，$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} -\frac{1}{2} & \frac{9}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，

$$\iiint_D \sin^2(x+3z)\cos^3(x-3y+z)dxdydz = \frac{1}{2}\int_{-\pi}^0 du \int_{-1}^1 dv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 u \cos^3 w dw$$
$$= \left(\int_{-\pi}^0 \frac{1-\cos 2u}{2} du\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 3w + 3\cos w}{4} dw\right)$$
$$= \frac{1}{8}\left[u - \frac{\sin 2u}{2}\right]_{-\pi}^0 \left[\frac{\sin 3w}{3} + 3\sin w\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3}\pi.$$

(3)　極座標変換より，

$$\iiint_D \sin\sqrt{x^2+y^2+z^2} dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{2\pi} r^2 \sin r \sin\theta dr$$
$$= 2\pi\left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} r^2 \sin r dr\right) = 2\pi[-\cos\theta]_0^{\pi}\left(\left[-r^2\cos r\right]_0^{2\pi} + 2\int_0^{2\pi} r\cos r dr\right)$$
$$= 4\left(-4\pi^2 + 2[r\sin r]_0^{2\pi} - 2\int_0^{2\pi} \sin r dr\right) = -16\pi^3.$$

(4)　対称性と極座標変換より，

$$\iiint_D |xyz| dxdydz = 8\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_2^3 r^5 \sin^3\theta \cos\theta \sin\varphi \cos\varphi dr$$

$$= 8\left(\int_2^3 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \frac{\sin 2\varphi}{2} d\varphi\right)$$
$$= 4\left[\frac{r^6}{6}\right]_2^3 \left[-\frac{\cos^4\theta}{4}\right]_0^{\frac{\pi}{2}} \left[-\frac{\cos 2\varphi}{2}\right]_0^{\frac{\pi}{2}} = \frac{665}{6}.$$

(5)　$x = 2r\sin\theta\cos\varphi$，$y = 3r\sin\theta\sin\varphi$，$z = 4r\cos\theta$ とおくと，

$$\iiint_D x^2z^2dxdydz = 24\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 64r^6\sin^3\theta\cos^2\theta\cos^2\varphi dr$$
$$= 1536\left(\int_0^1 r^6 dr\right)\left(\int_0^{\pi} \frac{2\sin\theta + \sin 3\theta - \sin 5\theta}{16} d\theta\right)\left(\int_0^{2\pi} \frac{1+\cos 2\varphi}{2} d\varphi\right)$$
$$= 48\left[\frac{r^7}{7}\right]_0^1 \left[-2\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5}\right]_0^{\pi} \left[\varphi + \frac{\sin 2\varphi}{2}\right]_0^{2\pi} = \frac{2048}{35}\pi.$$

(6)　$x = -\dfrac{u+3w}{2}$，$y = u+v+w$，$z = -\dfrac{u+w}{2}$ により，$E\colon 0 \leqq u \leqq 2,\ 0 \leqq v \leqq 1,\ 0 \leqq w \leqq 2$ は D に写り，$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} -\frac{1}{2} & 0 & -\frac{3}{2} \\ 1 & 1 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，

$$\iiint_D (x^2+y^2+z^2)dxdydz = \frac{1}{2}\int_0^1 du \int_0^1 dv \int_0^1 \left(\frac{3}{2}u^2 + v^2 + \frac{7}{2}w^2 + 2uv + 2vw + 4wu\right) dw$$
$$= \frac{1}{2}\int_0^1 du \int_0^1 \left(\frac{3}{2}u^2 + v^2 + \frac{7}{6} + 2uv + v + 2u\right) dv$$
$$= \frac{1}{2}\int_0^1 \left(\frac{3}{2}u^2 + \frac{1}{3} + \frac{7}{6} + u + \frac{1}{2} + 2u\right) du = \frac{1}{2}\int_0^1 \left(\frac{3}{2}u^2 + 3u + 2\right) du = 2.$$

(7)　極座標変換より，

$$\iiint_D \log(x^2+y^2+z^2)dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_1^2 2r^2\log r\sin\theta dr$$
$$= 4\pi\left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_1^2 r^2\log r dr\right) = 4\pi\left[-\cos\theta\right]_0^{\pi}\left(\left[\frac{r^3}{3}\log r\right]_1^2 - \frac{1}{3}\int_1^2 r^2 dr\right)$$
$$= \frac{8}{9}(24\log 2 - 7)\pi.$$

(8)　極座標変換より，

$$\iiint_D \frac{dxdydz}{x^2+y^2+(z-2)^2} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \frac{r^2\sin\theta}{r^2 - 4r\cos\theta + 4} dr$$
$$= 2\pi\int_0^1 \left[\frac{r}{4}\log\left|r^2 - 4r\cos\theta + 4\right|\right]_{\theta=0}^{\theta=\pi} dr = \pi\int_0^1 r\left\{\log(r+2) - \log(2-r)\right\} dr$$
$$= \pi\left(\int_0^1 \left\{(r+2)\log(r+2) - 2\log(r+2)\right\} + \int_0^1 \left\{(2-r)\log(2-r) - 2\log(2-r)\right\} dr\right)$$
$$= \pi\left(\left[\frac{1}{2}(r+2)^2\log(r+2)\right]_0^1 - \frac{1}{2}\int_0^1 (r+2)dr\right.$$
$$\left. - 2\left[(r+2)\log(r+2) - (r+2)\right]_0^1 - \left[\frac{1}{2}(2-r)^2\log(2-r)\right]_0^1\right.$$

$$
-\frac{1}{2}\int_0^1 (2-r)dr + 2\left[(2-r)\log(2-r)-(2-r)\right]_0^1 \Bigg)
$$

$$
= \frac{\pi}{2}(4-3\log 3).
$$

問 4

(1)　$D_n\colon \dfrac{1}{n} \leqq |x| \leqq 1,\ \dfrac{1}{n} \leqq |y| \leqq 1,\ \dfrac{1}{n} \leqq |z| \leqq 1$ とすると，対称性より，

$$
\iiint_D \frac{dxdydz}{|xyz|} = \lim_{n\to\infty}\iiint_{D_n} \frac{dxdydz}{|xyz|} = 8\left(\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \frac{dx}{x}\right)^3 = 8\left(\lim_{n\to\infty}\left[\log x\right]_{\frac{1}{n}}^1\right)^3 = \infty
$$

なので，発散する．

(2)　$D_n : 9 \leqq x^2+y^2+z^2 \leqq n^2$ とすると，極座標変換より，

$$
\iiint_D \frac{dxdydz}{(x^2+y^2+z^2)^2} = \lim_{n\to\infty}\iiint_{D_n} \frac{dxdydz}{(x^2+y^2+z^2)^2}
$$

$$
= \lim_{n\to\infty}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_3^n \frac{\sin\theta}{r^2}dr = 2\pi \lim_{n\to\infty}\left[-\cos\theta\right]_0^{\pi} \lim_{n\to\infty}\left[-\frac{1}{r}\right]_3^n = \frac{4}{3}\pi.
$$

(3)　$D_n : |x| \leqq n,\ |y| \leqq n,\ |z| \leqq n$ とすると，対称性より，

$$
\iiint_D x^2y^2z^2e^{-|x|-|y|-|z|}dxdydz = \lim_{n\to\infty}\iiint_{D_n} x^2y^2z^2e^{-|x|-|y|-|z|}dxdydz
$$

$$
= 8\lim_{n\to\infty}\left(\int_0^n x^2e^{-x}dx\right)^3 = 8\lim_{n\to\infty}\left(\left[-x^2e^{-x}\right]_0^n + 2\int_0^n xe^{-x}dx\right)^3
$$

$$
= 64\lim_{n\to\infty}\left(\left[-xe^{-x}\right]_0^n + \int_0^n e^{-x}dx\right)^3 = 64.
$$

(4)　$D_n\colon \dfrac{1}{n} \leqq x \leqq 1,\ \dfrac{1}{n} \leqq y \leqq 1,\ \dfrac{1}{n} \leqq z \leqq 1$ とすると，

$$
\iiint_D \frac{dxdydz}{(x+y+z)^2} = \lim_{n\to\infty}\iiint_{D_n} \frac{dxdydz}{(x+y+z)^2} = \lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1 dy\int_{\frac{1}{n}}^1 \frac{dz}{(x+y+z)^2}
$$

$$
= \lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1 \left[-\frac{1}{x+y+z}\right]_{z=\frac{1}{n}}^{z=1} dy
$$

$$
= \lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1 \left(-\frac{1}{x+y+1}+\frac{1}{x+y+\frac{1}{n}}\right) dy
$$

$$
= \lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left[-\log(x+y+1)+\log\left(x+y+\frac{1}{n}\right)\right]_{y=\frac{1}{n}}^{y=1} dx
$$

$$
= \lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left\{-\log(x+2)+2\log\left(x+1+\frac{1}{n}\right)-\log\left(x+\frac{2}{n}\right)\right\} dx
$$

$$
= \lim_{n\to\infty}\Bigg[-(x+2)\left\{\log(x+2)-1\right\}+2\left(x+1+\frac{1}{n}\right)\left\{\log\left(x+1+\frac{1}{n}\right)-1\right\}
$$

$$
-\left(x+\frac{2}{n}\right)\left\{\log\left(x+\frac{2}{n}\right)-1\right\}\Bigg]_{\frac{1}{n}}^1 = 6\log 2-3\log 3.
$$

(5)　$D_n\colon -n \leqq x+y+z \leqq 1,\ |x+y+z+1| \geqq \dfrac{1}{n},\ |y| \leqq n,\ |z| \leqq n$ とし，$x=u-v-w$，$y=v$，

$z=w$ と変換して，

$$\iiint_D \frac{dxdydz}{|x+y+z+1|^3} = \lim_{n\to\infty}\iiint_{D_n} \frac{dxdydz}{|x+y+z+1|^3}$$
$$\geqq 4\lim_{n\to\infty} n^2 \int_{-n}^{-1-\frac{1}{n}} \frac{du}{(-u-1)^3} = 4\lim_{n\to\infty} n^3 \left[\frac{1}{2}(-u-1)^{-2}\right]_{-n}^{-1-\frac{1}{n}} = \infty$$

より，発散する．

(6)　$D_n\colon \dfrac{1}{n^2} \leqq x^2+y^2+z^2 \leqq 1$ とし，極座標変換より，

$$\iiint_D \log(x^2+y^2+z^2)dxdydz = \lim_{n\to\infty}\iiint_{D_n} \log(x^2+y^2+z^2)dxdydz$$
$$= \lim_{n\to\infty}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{\frac{1}{n}}^{1} \left(\log r^2\right) r^2 \sin\theta dr = 4\pi \lim_{n\to\infty}\left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_{\frac{1}{n}}^{1} r\log r dr\right)$$
$$= 4\pi\left[-\cos\theta\right]_0^{\pi} \lim_{n\to\infty}\left(\left[\frac{r^3}{3}\log r\right]_{\frac{1}{n}}^{1} - \frac{1}{3}\int_{\frac{1}{n}}^{1} r^2 dr\right) = \frac{8}{3}\pi \lim_{n\to\infty}\left[-\frac{r^3}{3}\right]_{\frac{1}{n}}^{1} = -\frac{8}{9}\pi.$$

(7)　$D_n\colon \dfrac{1}{n^2} \leqq x^2+y^2+z^2 \leqq 1$ とし，極座標変換より，

$$\iiint_D \frac{dxdydz}{|x^3+y^4+z^5|} = \lim_{n\to\infty}\iiint_{D_n} \frac{dxdydz}{|x^3+y^4+z^5|}$$
$$\geqq \lim_{n\to\infty}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{n}}^{1} \frac{\sin\theta}{r(\sin^3\theta\cos^3\varphi + \sin^3\sin^3\varphi + \cos^3\theta)} dr$$
$$\geqq \frac{\pi}{2}\lim_{n\to\infty}\int_{\frac{1}{n}}^{1} \frac{dr}{r} = \infty$$

なので，発散する．

(8)　$D_n\colon \dfrac{1}{n^2} \leqq x^2+y^2+z^2 \leqq 1$ とし，極座標変換より，

$$\iiint_D \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}} = \lim_{n\to\infty}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{\frac{1}{n}}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}} dr$$
$$= 4\pi\lim_{n\to\infty}\int_{\frac{1}{n}}^{1}\left(-\sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}\right)dx = 2\pi\lim_{n\to\infty}\left[\sin^{-1}x - x\sqrt{1-x^2}\right]_{\frac{1}{n}}^{1} = \pi^2.$$

(9)　$D_n\colon x^2+y^2+z^2 \leqq n^2,\ x \geqq 0, y \geqq 0,\ z \geqq 0$ とすると，極座標変換より，

$$\iiint_D \frac{xyz}{(1+x^2+y^2+z^2)^4} dxdydz = \lim_{n\to\infty}\iiint_{D_n} \frac{xyz}{(1+x^2+y^2+z^2)^4} dxdydz$$
$$= \lim_{n\to\infty}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{n} \frac{r^5\sin^3\theta\cos\theta\sin\varphi\cos\varphi}{(1+r^2)^4} dr$$
$$= \left(\int_0^{\frac{\pi}{2}} \frac{\sin 2\varphi}{2} d\varphi\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta d\theta\right)\lim_{n\to\infty}\int_0^{n} \frac{r^5}{(1+r^2)^4} dr$$
$$= \left[-\frac{\cos 2\varphi}{4}\right]_0^{\frac{\pi}{2}} \left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{2}} \lim_{n\to\infty}\left[-\frac{3r^4+3r^2+1}{6(r^2+1)^3}\right]_0^{n} = \frac{1}{48}.$$

(10)　$x=u-v$，$y=v$，$z=w$ により，$E_n\colon u^2+v^2+w^2 \leqq n^2,\ u \geqq 0$ は $D_n\colon (x+y)^2+y^2+z^2 \leqq n^2,\ x+y \geqq 0$ に写るので，極座標変換より，

$$\iiint_D (x+y)e^{-(x^2+2xy+2y^2+z^2)} dxdydz = \lim_{n\to\infty}\iiint_{D_n} (x+y)e^{-(x^2+2xy+2y^2+z^2)} dxdydz$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \iiint_{E_n} ue^{-(u^2+v^2+w^2)} dudvdw = \lim_{n\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\pi} d\theta \int_0^n r^3 e^{-r^2} \sin^2\theta \cos\varphi dr \\
&= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi d\varphi \right) \left(\int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta \right) \lim_{n\to\infty} \int_0^n r^3 e^{-r^2} dr \\
&= \frac{1}{2} \left[\sin\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\theta - \frac{1}{2}\sin 2\theta \right]_0^{\pi} \lim_{n\to\infty} \left(\left[-\frac{r^2}{2} e^{-r^2} \right]_0^n + \int_0^n re^{-r^2} dr \right) \\
&= \pi \lim_{n\to\infty} \left[-\frac{1}{2} e^{-r^2} \right]_0^n = \frac{\pi}{2}.
\end{aligned}
$$