5　重積分

5.3　広義 2 重積分

問 1 $J=\iint_D e^{-x^2-y^2}x^{2p-1}y^{2q-1}dxdy$ とおく．近似列 $D_n=\{(x,y)\mid x^2+y^2\leqq n^2,\ x\geqq 0,\ y\geqq 0\}$ では，$s=r^2$ とすると $\dfrac{dr}{ds}=\dfrac{1}{2\sqrt{s}}$ なので，

$$
\begin{aligned}
J&=\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^n e^{-r^2}r^{2(p+q)-1}\cos^{2p-1}\theta\sin^{2q-1}\theta dr\\
&=\frac{1}{2}\left(\int_0^\infty e^{-s}s^{-p-q-1}ds\right)\left(\int_0^{\frac{\pi}{2}}\cos^{2p-1}\theta\sin^{2q-1}d\theta\right)=\frac{1}{4}\Gamma(p+q)B(p,q).
\end{aligned}
$$

一方，$E_n=\{(x,y)\mid 0\leqq x\leqq n,\ 0\leqq y\leqq n\}$ では，$u=x^2$，$v=y^2$ とすると $\dfrac{dx}{du}=\dfrac{1}{2\sqrt{u}}$，$\dfrac{dy}{dv}=\dfrac{1}{2\sqrt{v}}$ なので，

$$
J=\lim_{n\to\infty}\left(\int_0^n e^{-x^2}x^{2p-1}dx\right)\left(\int_0^n e^{-y^2}y^{2q-1}dy\right)=\frac{1}{4}\Gamma(p)\Gamma(q).
$$

よって，$B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ である．

問 2

(1)　$D_n\colon 0\leqq x\leqq n,\ 0\leqq y\leqq n$ とすると，

$$
\begin{aligned}
&\iint_D\frac{dxdy}{(x+y+1)^5}=\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{(x+y+1)^5}=\lim_{n\to\infty}\int_0^n dx\int_0^n\frac{dy}{(x+y+1)^5}\\
&=\lim_{n\to\infty}\int_0^n\left[-\frac{1}{4}(x+y+1)^{-4}\right]_{y=0}^{y=n}dx\\
&=-\frac{1}{4}\lim_{n\to\infty}\int_0^n\left\{(x+n+1)^{-4}-(x+1)^{-4}\right\}dx\\
&=-\frac{1}{4}\lim_{n\to\infty}\left[-\frac{1}{3}\left\{(x+n+1)^{-3}-(x+1)^{-3}\right\}\right]_0^n=\frac{1}{12}.
\end{aligned}
$$

(2)　$D_n\colon x^2+y^2\leqq n^2$ とし，$x=r\cos\theta$，$y=r\sin\theta$ とおくと，

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}\frac{dxdy}{(x^2+y^2+1)^{\frac{3}{2}}}=\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{(x^2+y^2+1)^{\frac{3}{2}}}\\
&=\lim_{n\to\infty}\int_0^n dr\int_0^{2\pi}\frac{r}{(r^2+1)^{\frac{3}{2}}}d\theta=2\pi\lim_{n\to\infty}\int_0^n\left[-\left(r^2+1\right)^{-\frac{1}{2}}\right]_0^n=2\pi.
\end{aligned}
$$

(3)　$D_n\colon x^2+y^2\leqq 4-\dfrac{1}{n}$ とし，$x=r\cos\theta$，$y=r\sin\theta$ とおくと，

$$
\begin{aligned}
&\iint_D\frac{dxdy}{\sqrt{4-x^2-y^2}}=\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{\sqrt{4-x^2-y^2}}\\
&=\lim_{n\to\infty}\int_0^{\sqrt{4-\frac{1}{n}}}dr\int_0^{2\pi}\frac{r}{\sqrt{4-r^2}}d\theta=2\pi\lim_{n\to\infty}\left[-\sqrt{4-r^2}\right]_0^{\sqrt{4-\frac{1}{n}}}=4\pi.
\end{aligned}
$$

(4)　$D_n\colon x^2+y^2\leqq n^2,\ x\geqq 0,\ y\geqq 0$ とし，$x=r\cos\theta$，$y=r\sin\theta$ とおくと，対称性より，

$$
\iint_D e^{-x^2-y^2}dxdy=\lim_{n\to\infty}\iint_{D_n}e^{-x^2-y^2}dxdy=2\lim_{n\to\infty}\int_0^n dr\int_0^{\frac{\pi}{2}}e^{-r^2}rd\theta
$$

$$= \pi \lim_{n\to\infty} \left[-\frac{1}{2} e^{-r^2} \right]_0^n = \frac{\pi}{2}.$$

(5)　$D_n \colon 1 \leqq |x| \leqq n,\, 1 \leqq |y| \leqq n$ とすると，対称性より，

$$\iint_D \frac{dxdy}{(|x|+|y|)^4} = \lim_{n\to\infty} \iint_{D_n} \frac{dxdy}{(|x|+|y|)^4} = 4 \lim_{n\to\infty} \int_1^n dx \int_1^n \frac{dy}{(x+y)^4}$$
$$= 4 \lim_{n\to\infty} \int_1^n \left[-\frac{1}{3}(x+y)^{-3} \right]_{y=1}^{y=n} dx = \frac{4}{3} \lim_{n\to\infty} \int_1^n \left\{ (x+1)^{-3} - (x+n)^{-3} \right\} dx$$
$$= \frac{4}{3} \lim_{n\to\infty} \left[-\frac{1}{2} \left\{ (x+1)^{-2} - (x+n)^{-2} \right\} \right]_1^n = \frac{1}{6}.$$

(6)　$D_n \colon y + \dfrac{1}{n} \leqq x \leqq \pi - \dfrac{1}{n},\, 0 \leqq y \leqq \pi$ とすると，

$$\iint_D \frac{\sin y}{\sqrt{(\pi - x)(x-y)}} dxdy = \lim_{n\to\infty} \iint_{D_n} \frac{\sin y}{\sqrt{(\pi - x)(x-y)}} dxdy$$
$$= \lim_{n\to\infty} \int_0^\pi dy \int_{y+\frac{1}{n}}^{\pi - \frac{1}{n}} \frac{\sin y}{\sqrt{\left(\frac{\pi - y}{2}\right)^2 - \left(x - \frac{y+\pi}{2}\right)^2}} dx = \lim_{n\to\infty} \int_0^\pi \sin y \left[\sin^{-1} \frac{2x - \pi - y}{\pi - y} \right]_{x=y+\frac{1}{n}}^{x=\pi-\frac{1}{n}} dy$$
$$= \pi \int_0^\pi \sin y dy = 2\pi.$$

(7)　$x = \dfrac{2u+v}{5}$，$y = \dfrac{-u+2v}{5}$ とおくと，$E_n \colon \dfrac{1}{n} \leqq v \leqq 2,\, -\dfrac{v}{2} \leqq u \leqq 2v$ は $D_n \colon \dfrac{1}{n} \leqq x + 2y \leqq 2,\, x \geqq 0,\, y \geqq 0$ に写り，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} = \dfrac{1}{5}$ なので，

$$\iint_D \exp\left(\frac{2x - y}{x + 2y} \right) dxdy = \lim_{n\to\infty} \frac{1}{5} \int_{\frac{1}{n}}^2 dv \int_{-\frac{v}{2}}^{2v} e^{\frac{u}{v}} du$$
$$= \lim_{n\to\infty} \frac{1}{5} \int_{\frac{1}{n}}^2 \left[v e^{\frac{u}{v}} \right]_{-\frac{v}{2}}^{2v} du = \frac{1}{5} \left(e^2 - e^{-\frac{1}{2}} \right) \int_0^2 v dv = \frac{2}{5} \left(e^2 - e^{-\frac{1}{2}} \right).$$

(8)　$x = r\cos\theta$，$y = r\sin\theta$ により，$D_n \colon x^2 + y^2 \geqq \dfrac{1}{n^2},\, 0 \leqq x \leqq y \leqq 1$ は $E_n \colon \dfrac{1}{n} \leqq r \leqq \dfrac{1}{\sin\theta},\, \dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{2}$ に写る．対称性より，

$$\iint_D \frac{dxdy}{\sqrt{x^2+y^2}} = 4 \lim_{n\to\infty} \iint_{D_n} \frac{dxdy}{\sqrt{x^2+y^2}} = 4 \lim_{n\to\infty} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_{\frac{1}{n}}^{\frac{1}{\sin\theta}} dr = 4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}.$$

ここで，$t = \tan\dfrac{x}{2}$ とすると，$\tan\dfrac{\pi}{8} = \dfrac{-1+\sqrt{1+\tan\frac{\pi}{4}}}{\tan\frac{\pi}{4}} = \sqrt{2} - 1$ より，

$$\iint_D \frac{dxdy}{\sqrt{x^2+y^2}} = 4 \int_{\sqrt{2}-1}^1 \frac{dt}{t} = 4 \left[\log t \right]_{\sqrt{2}-1}^1 = 4 \log\left(\sqrt{2} + 1 \right).$$

問 3　求める極限を I とおく．

(1)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^\infty D_n = \{(x,y) \mid 0 < x \leqq 1,\, 0 < y \leqq 1\}$ なので，D の近似列である．$\dfrac{x^2 - y^2}{(x^2+y^2)^2} = -\dfrac{1}{x^2+y^2} + \dfrac{2x^2}{(x^2+y^2)^2} = \dfrac{\partial}{\partial x}\left(-\dfrac{x}{x^2+y^2} \right)$ なので，

$$I = \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \left[-\frac{x}{x^2+y^2} \right]_{\frac{1}{n}}^1 dy = \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \left(-\frac{1}{1+y^2} + \frac{1}{n} \frac{1}{y^2 + \frac{1}{n^2}} \right) dy$$

$$= \lim_{n\to\infty} \left[-\tan^{-1} y + \tan^{-1} ny \right]_{\frac{1}{n}}^{1} = 0.$$

(2)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y) \mid 0 < x \leqq 1,\, 0 < y \leqq 1\}$ なので，D の近似列である．

$$\begin{aligned} I &= \lim_{n\to\infty} \int_{\frac{\sqrt{3}}{n}}^{1} \left[-\frac{x}{x^2+y^2} \right]_{\frac{1}{n}}^{1} dy = \lim_{n\to\infty} \int_{\frac{\sqrt{3}}{n}}^{1} \left(-\frac{1}{1+y^2} + \frac{1}{n}\frac{1}{y^2+\frac{1}{n^2}} \right) dy \\ &= \lim_{n\to\infty} \left[-\tan^{-1} y + \tan^{-1} ny \right]_{\frac{\sqrt{3}}{n}}^{1} = -\frac{\pi}{12}. \end{aligned}$$

(3)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y) \mid 0 \leqq x \leqq 1,\, 0 < y \leqq 1\}$ なので，D の近似列である．$x = r\cos\theta$，$y = r\sin\theta$ とすると，$E_n\colon n^{-\frac{n}{2}} \leqq r \leqq \min\left(\dfrac{1}{\cos\theta}, \dfrac{1}{\sin\theta}\right)$，$\dfrac{1}{n} \leqq \theta \leqq \dfrac{\pi}{2}$ は D_n に写るので，

$$\begin{aligned} I &= \lim_{n\to\infty} \left(\int_{\frac{1}{n}}^{\frac{\pi}{4}} d\theta \int_{n^{-\frac{n}{2}}}^{\frac{1}{\cos\theta}} \frac{\cos 2\theta}{r} dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_{n^{-\frac{n}{2}}}^{\frac{1}{\sin\theta}} \frac{\cos 2\theta}{r} dr \right) \\ &= \int_0^{\frac{\pi}{4}} \cos 2\theta \log \frac{1}{\cos\theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \log \frac{1}{\sin\theta} d\theta \\ &\quad + \lim_{n\to\infty} \frac{n}{2} \log n \int_{\frac{1}{n}}^{\frac{\pi}{2}} \cos 2\theta d\theta \quad \left(\text{第 2 項では } \theta' = -\theta + \frac{\pi}{2} \text{ と置換する} \right) \\ &= -\frac{1}{2} \lim_{n\to\infty} \frac{\sin\frac{2}{n}}{\frac{2}{n}} \log n = -\infty. \end{aligned}$$

(4)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y) \mid 0 < x \leqq 1,\, 0 \leqq y \leqq 1\}$ なので，D の近似列である．$x = r\cos\theta$，$y = r\sin\theta$ とすると，$E_n\colon n^{-\frac{n}{2}} \leqq r \leqq \min\left(\dfrac{1}{\cos\theta}, \dfrac{1}{\sin\theta}\right)$, $0 \leqq \theta \leqq \dfrac{\pi}{2} - \dfrac{1}{n}$ は D_n は写るので，

$$\begin{aligned} I &= \lim_{n\to\infty} \left(\int_0^{\frac{\pi}{4}} d\theta \int_{n^{-\frac{n}{2}}}^{\frac{1}{\cos\theta}} \frac{\cos 2\theta}{r} dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{1}{n}} d\theta \int_{n^{-\frac{n}{2}}}^{\frac{1}{\cos\theta}} \frac{\cos 2\theta}{r} dr \right) \\ &= \int_0^{\frac{\pi}{4}} \cos 2\theta \log \frac{1}{\cos\theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \log \frac{1}{\sin\theta} d\theta + \lim_{n\to\infty} \frac{n}{2} \log n \int_0^{\frac{\pi}{2}-\frac{1}{n}} \cos 2\theta d\theta \\ &= \frac{1}{2} \lim_{n\to\infty} \frac{\sin\frac{2}{n}}{\frac{2}{n}} \log n = +\infty. \end{aligned}$$

問 4

(1)　$x = r\cos\theta$，$y = r\sin\theta$ とし，$D_n\colon x^2 + y^2 \leqq n^2$ とすると，

$$\iint_{\mathbb{R}^2} \frac{dxdy}{1+(x^2+y^2)^2} = 2\pi \lim_{n\to\infty} \int_0^n \frac{r}{1+r^4} dr = 2\pi \lim_{n\to\infty} \left[\frac{1}{2} \tan^{-1} r^2 \right]_0^n = \frac{\pi^2}{2}.$$

(2)　$D_n\colon 0 \leqq x \leqq n,\, 0 \leqq y \leqq n$ とすると，

$$\iint_D ye^{-xy} dxdy = \lim_{n\to\infty} \int_0^n dy \int_0^n ye^{-xy} dx = \lim_{n\to\infty} \int_0^n \left[-e^{-xy} \right]_{x=0}^{x=n} dy$$

$$= \lim_{n\to\infty}\int_0^n \left(1-e^{-ny}\right)dy = \lim_{n\to\infty}\left[y+\frac{1}{n}e^{-ny}\right]_0^n = \infty$$

なので，発散する．

(3)　$x = r\cos\theta$，$y = r\sin\theta$ とし，$D_n\colon \dfrac{1}{n^2} \leqq x^2+y^2 \leqq 1$ とすると，

$$\iint_D \frac{|xy|}{x^4+y^4}dxdy = \lim_{n\to\infty}\int_{\frac{1}{n}}^1 dr\int_0^{2\pi}\frac{|\cos\theta\sin\theta|}{r(\cos^4\theta+\sin^4\theta)}d\theta$$
$$= \int_0^{2\pi}\frac{|\cos\theta\sin\theta|}{\cos^4\theta+\sin^4\theta}d\theta \lim_{n\to\infty}\int_{\frac{1}{n}}^1\frac{dr}{r} \geqq \lim_{n\to\infty}\left[\log r\right]_{\frac{1}{n}}^1 = \infty$$

より，発散する．

(4)　$D_n\colon \dfrac{1}{n} \leqq ||x|-|y|| \leqq n$ とすると，対称性より，

$$\iint_{\mathbb{R}^2}\frac{dxdy}{|x^2-y^2|} = \lim_{n\to\infty}\frac{dxdy}{|x^2-y^2|} = 4\lim_{n\to\infty}\int_{\frac{1}{n}}^n dx\int_0^{x-\frac{1}{n}}\frac{dy}{x^2-y^2}$$
$$= 4\lim_{n\to\infty}\int_{\frac{1}{n}}^n\frac{1}{2x}\left[-\log(x-y)+\log(x+y)\right]_{y=0}^{y=x-\frac{1}{n}}dx$$
$$= 2\lim_{n\to\infty}\int_{\frac{1}{n}}^n\frac{1}{x}\log(2nx-1)\,dx \geqq \lim_{n\to\infty}\int_1^n\frac{1}{x}\log(2nx-1)dx$$
$$\geqq \lim_{n\to\infty}\log n\int_1^n\frac{dx}{x} = \lim_{n\to\infty}(\log n)^2 = \infty$$

より，発散する．

(5)　$x = \dfrac{u+v}{2}$，$y = \dfrac{u-v}{2}$ により，$E_n\colon |u| \leqq n,\ |v| \leqq n$ は $D_n\colon |x|+|y| \leqq n$ に写り，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix} = -\dfrac{1}{2}$ なので，対称性より，

$$\iint_{\mathbb{R}^2}|x^2-y^2|e^{-|x|-|y|}dxdy = \lim_{n\to\infty}\iint_{D_n}|x^2-y^2|e^{-|x|-|y|}dxdy$$
$$= 4\lim_{n\to\infty}\int_0^n du\int_0^u uve^{-u}dv = 2\lim_{n\to\infty}\int_0^n u^3e^{-u}du$$
$$= 2\lim_{n\to\infty}\left(\left[-u^3e^{-u}\right]_0^n + 3\int_0^n u^2e^{-u}du\right)$$
$$= 6\lim_{n\to\infty}\left(\left[-u^2e^{-u}\right]_0^n + 2\int_0^n ue^{-u}du\right)$$
$$= 12\lim_{n\to\infty}\left(\left[-ue^{-u}\right]_0^n + \int_0^n e^{-u}du\right) = 12\lim_{n\to\infty}\left[-e^{-u}\right]_0^n = 12.$$

(6)　$E\colon \dfrac{\pi}{4} \leqq x+y \leqq \dfrac{3}{4}\pi$ とすると，$E_n\colon -n \leqq x \leqq n,\ \dfrac{\pi}{4} \leqq x+y \leqq \dfrac{3}{4}\pi$ は E の近似列で，

$$\int_{\mathbb{R}^2}\frac{dxdy}{|\sin(x+y)|} \geqq \int_E\frac{dxdy}{|\sin(x+y)|} \geqq \iint_E dxdy = \lim_{n\to\infty}\int_{-n}^n dx\int_{-x+\frac{\pi}{4}}^{-x+\frac{3}{4}\pi}dy = \frac{\pi}{2}\lim_{n\to\infty}2n = \infty$$

より，発散する．

(7)　$D_n\colon \dfrac{1}{n^2} \leqq x^2+y^2 \leqq \dfrac{1}{4}$ とすると，極座標変換より，

$$\iint_D\frac{dxdy}{(x^2+y^2)\{\log(x^2+y^2)\}^2} = \lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{(x^2+y^2)\{\log(x^2+y^2)\}^2}$$

$$= \frac{\pi}{2} \lim_{n\to\infty} \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{1}{r(\log r)^2} = \frac{\pi}{2} \lim_{n\to\infty} \left[-\frac{1}{\log r} \right]_{\frac{1}{n}}^{\frac{1}{2}} = \frac{\pi}{2\log 2}.$$

(8) $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$ により，$E_n\colon \dfrac{1}{n} \leqq |u| \leqq 1,\ 0 \leqq u+v \leqq 2$ は $D_n\colon 0 \leqq x \leqq 1,\ \dfrac{1}{n} \leqq |x+y| \leqq 1$ に写り，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，

$$\iint_D \frac{|x-y|}{\sqrt{|x+y|}} dxdy = \lim_{n\to\infty} \iint_{D_n} \frac{|x-y|}{\sqrt{|x+y|}} dxdy = \frac{1}{2} \lim_{n\to\infty} \iint_{E_n} \frac{|v|}{\sqrt{|u|}} dudv$$
$$= \frac{1}{2} \lim_{n\to\infty} \left(\int_{\frac{1}{n}}^{1} du \int_{-u}^{0} \frac{-v}{\sqrt{u}} dv + \int_{\frac{1}{n}}^{1} du \int_{0}^{2-u} \frac{v}{\sqrt{u}} dv + \int_{-1}^{-\frac{1}{n}} du \int_{-u}^{2-u} \frac{v}{\sqrt{-u}} dv \right)$$
$$= \frac{1}{4} \lim_{n\to\infty} \left\{ \int_{\frac{1}{n}}^{1} u^{\frac{3}{2}} du + \int_{\frac{1}{n}}^{1} \left(4u^{-\frac{1}{2}} - 4u^{\frac{1}{2}} + u^{\frac{3}{2}} \right) du + \int_{-1}^{-\frac{1}{n}} \left(\frac{4}{\sqrt{-u}} + 4\sqrt{-u} \right) du \right\}$$
$$= \frac{1}{4} \lim_{n\to\infty} \left(\left[\frac{5}{2} u^{\frac{5}{2}} \right]_{\frac{1}{n}}^{1} + \left[8u^{\frac{1}{2}} - \frac{8}{3} u^{\frac{3}{2}} \right]_{\frac{1}{n}}^{1} + \left[-8\sqrt{-u} - \frac{8}{3}(-u)^{\frac{3}{2}} \right]_{-1}^{-\frac{1}{n}} \right) = \frac{37}{4} < \infty$$

より収束する．同様の計算により，

$$\iint_D \frac{x-y}{\sqrt{|x+y|}} dxdy$$
$$= \frac{1}{2} \lim_{n\to\infty} \left(\int_{\frac{1}{n}}^{1} du \int_{-u}^{0} \frac{v}{\sqrt{u}} dv + \int_{\frac{1}{n}}^{1} du \int_{0}^{2-u} \frac{v}{\sqrt{u}} dv + \int_{-1}^{-\frac{1}{n}} du \int_{-u}^{2-u} \frac{v}{\sqrt{-u}} dv \right)$$
$$= \frac{1}{4} \lim_{n\to\infty} \left(-\left[\frac{5}{2} u^{\frac{5}{2}} \right]_{\frac{1}{n}}^{1} + \left[8u^{\frac{1}{2}} - \frac{8}{3} u^{\frac{3}{2}} \right]_{\frac{1}{n}}^{1} + \left[-8\sqrt{-u} - \frac{8}{3}(-u)^{\frac{3}{2}} \right]_{-1}^{-\frac{1}{n}} \right) = 4.$$

(9) $D_n\colon 1 \leqq |x| \leqq n,\ |x-y| \leqq 1$ とすると，

$$\iint_D xye^{-|x|-|y|} dxdy$$
$$= \lim_{n\to\infty} \left(\int_1^n dx \int_{x-1}^{x+1} xe^{-x} ye^{-y} dy + \int_{-n}^{-1} dx \int_{x-1}^{x+1} xe^x ye^y dy \right)$$
$$= \lim_{n\to\infty} \left(\int_1^n xe^{-x} \left[-(y+1)e^{-y} \right]_{x-1}^{x+1} dx + \int_{-n}^{-1} xe^x \left[(y-1)e^y \right]_{x-1}^{x+1} dx \right)$$
$$= 2 \lim_{n\to\infty} \int_1^n \left\{ \left(e - \frac{1}{e} \right) x^2 - \frac{2}{e} x \right\} e^{-2x} dx$$
$$= 2 \lim_{n\to\infty} \left(\left[-\frac{1}{2} \left\{ \left(e - \frac{1}{e} \right) x^2 - \frac{2}{e} x \right\} e^{-2x} \right]_1^n + \int_1^n \left\{ \left(e - \frac{1}{e} \right) x - \frac{1}{e} \right\} e^{-2x} dx \right)$$
$$= \frac{1}{e} - \frac{3}{e^3} + 2 \lim_{n\to\infty} \left(\left[-\frac{1}{2} \left\{ \left(e - \frac{1}{e} \right) x - \frac{1}{e} \right\} e^{-2x} \right]_1^n + \frac{1}{2} \left(e - \frac{1}{e} \right) \int_1^n e^{-2x} dx \right)$$
$$= \frac{2}{e} - \frac{5}{e^3} + \left(e - \frac{1}{e} \right) \lim_{n\to\infty} \left[-\frac{1}{2} e^{-2x} \right]_1^n = \frac{1}{2} \left(\frac{5}{e} - \frac{11}{e^3} \right).$$

(10) $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$ とおくと，$E_n\colon |uv| \geqq 1,\ |u| \leqq n,\ |v| \leqq n$ は $D_n\colon |x^2-y^2| \geqq 1,\ |x+y| \leqq n,\ |x-y| \leqq n$ に写り，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，対称性より，

$$\iint_D \frac{dxdy}{|x-y||\log|x+y||} = \lim_{n\to\infty} \iint_{D_n} \frac{dxdy}{|x-y||\log|x+y||}$$

$$
\begin{aligned}
&= 2\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} du \int_{\frac{1}{u}}^{n} \frac{dv}{v|\log u|} = 2\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} \frac{1}{|\log u|}\left(\log n + |\log u|\right) du \\
&\geqq 2\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} du = \infty
\end{aligned}
$$

となり，発散する．

(11)　$x = \dfrac{1}{2} r\cos\theta$，$y = \dfrac{1}{\sqrt{3}} r\sin\theta$ により，$E_n\colon 0 \leqq r \leqq \dfrac{1}{n}$，$\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{3}{2}\pi$ は $D_n\colon x \leqq 0$，$\dfrac{1}{n^2} \leqq 4x^2 + 3y^2 \leqq 1$ に写り，$\dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{r}{2\sqrt{3}}$ なので，

$$
\begin{aligned}
&\iint_D \frac{|x|}{3x^2+4y^2} dxdy = \lim_{n\to\infty}\iint_{D_n} \frac{-x}{3x^2+4y^2} dxdy \\
&= \frac{6}{\sqrt{3}} \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\theta \int_0^1 \frac{-r\cos\theta}{9\cos^2\theta + 16\sin^2\theta} dr = \sqrt{3}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{-\cos\theta}{7\sin^2\theta+9} d\theta \\
&= \sqrt{3}\left[-\frac{1}{3\sqrt{7}}\tan^{-1}\frac{\sqrt{7}\sin\theta}{3}\right]_{\frac{\pi}{2}}^{\frac{3}{2}\pi} = \frac{2}{\sqrt{21}}\tan^{-1}\frac{\sqrt{7}}{3} < \infty.
\end{aligned}
$$

従って，広義積分は収束し，求める値は $-\dfrac{2}{\sqrt{21}}\tan^{-1}\dfrac{\sqrt{7}}{3}$.