5.2　変数変換

問 1

(1)　$u=x+2y$，$v=3x-y$ とおくと $x=\dfrac{u+2v}{7}$，$y=\dfrac{3u-v}{7}$ なので，$\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{7} & \frac{2}{7}\\ \frac{3}{7} & -\frac{1}{7}\end{pmatrix}$ $=-\dfrac{1}{7}$．よって，

$$\iint_D \cos(x+2y)\sin(3x-y)dxy=\frac{1}{7}\left(\int_{\frac{\pi}{2}}^{\pi}\cos u du\right)\left(\int_{-\frac{\pi}{2}}^{0}\sin v dv\right)$$
$$=\frac{1}{7}\left[\sin u\right]_{\frac{\pi}{2}}^{\pi}\left[-\cos v\right]_{-\frac{\pi}{2}}^{0}=\frac{1}{7}.$$

(2)　$x=r\cos\theta$，$y=r\sin\theta$ と変換して，

$$\iint_D (x^2+y^2)^2dxdy=2\pi\int_0^2 r^5dr=2\pi\left[\frac{r^6}{6}\right]_0^2=\frac{64}{3}\pi.$$

(3)　$u=x+y$，$v=x-y$ とおくと $x=\dfrac{u+v}{2}$，$y=\dfrac{u-v}{2}$ なので，$\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=$ $-\dfrac{1}{2}$．よって，

$$\iint_D (x+y)^2|\cos\pi(x-y)|dxdy=\frac{1}{2}\left(\int_{-3}^{3}u^2du\right)\left(\int_{-3}^{3}|\cos\pi v|dv\right)$$
$$=12\left(\int_0^3 u^2du\right)\left(\int_0^{\frac{1}{2}}\cos\pi v dv\right)=12\left[\frac{u^3}{3}\right]_0^3\left[\frac{\sin\pi v}{\pi}\right]_0^{\frac{1}{2}}=\frac{108}{\pi}.$$

(4)　$u=x+y$，$v=x-y$ とおくと $x=\dfrac{u+v}{2}$，$y=\dfrac{u-v}{2}$ なので $\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=$ $-\dfrac{1}{2}$．よって，

$$\iint_D (x^2-y^2)^4dxdy=\frac{1}{2}\left(\int_{-2}^{2}u^4du\right)\left(\int_{-2}^{2}v^4dv\right)=2\left(\int_0^2 u^4du\right)^2=2\left(\left[\frac{u^5}{5}\right]_0^2\right)^2=\frac{2048}{25}.$$

(5)　$x=r\cos\theta$，$y=r\sin\theta$ と変換して，

$$\iint_D e^{x^2+y^2}dxdy=2\pi\int_2^7 e^{r^2}rdr=2\pi\left[\frac{e^{r^2}}{2}\right]_2^7=\pi\left(e^{49}-e^4\right).$$

(6)　$x=r\cos\theta$，$y=r\sin\theta$ と変換して，

$$\iint_D \frac{dxdy}{\sqrt{x^2+y^2+1}}=2\pi\int_1^{\sqrt{5}}\frac{r}{\sqrt{r^2+1}}dr=2\pi\left[\sqrt{r^2+1}\right]_1^{\sqrt{5}}=2\pi\left(\sqrt{6}-\sqrt{2}\right).$$

(7)　$x=3r\cos\theta$，$y=2r\sin\theta$ とおくと $\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}3\cos\theta & -3r\sin\theta\\ 2\sin\theta & 2r\cos\theta\end{pmatrix}=6r$ なので，

$$\iint_D x^2y^2dxdy=6\int_0^{2\pi}d\theta\int_0^1 r^5\cos^2\theta\sin^2\theta dr=216\left(\int_0^{2\pi}\frac{\sin^2 2\theta}{4}d\theta\right)\left(\int_0^1 r^5dr\right)$$

$$= 216\int_0^{2\pi} \frac{1-\cos 4\theta}{8} d\theta \left[\frac{r^6}{6}\right]_0^1 = \frac{9}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = 9\pi.$$

(8) $u = x+y$，$v = x-y$ とおくと $x = \dfrac{u+v}{2}$，$y = \dfrac{u-v}{2}$ なので，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$．よって，

$$\iint_D \frac{(x-y)^2}{(1+x+y)^5} dxdy = \frac{1}{2}\int_0^1 du \int_{-u}^{u} \frac{v^2}{(1+u)^5} dv = \int_0^1 du \int_0^u \frac{v^2}{(1+u)^5} dv$$
$$= \int_0^1 \frac{1}{(1+u)^5}\left[\frac{v^3}{3}\right]_0^u du = \frac{1}{3}\int_0^1 \frac{u^3}{(u+1)^5} du.$$

$t = u+1$ と置換して，

$$\frac{1}{3}\int_0^1 \frac{u^3}{(u+1)^5} du = \frac{1}{3}\int_1^2 \frac{(t-1)^3}{t^5} dt = \frac{1}{3}\int_1^2 \left(t^{-2} - 3t^{-3} + 3t^{-4} - t^{-5}\right) dt$$
$$= \frac{1}{3}\left[-t^{-1} + \frac{3}{2}t^{-2} - t^{-3} + \frac{1}{4}t^{-4}\right]_1^2 = \frac{1}{192}.$$

(9) $x = r\cos\theta$，$y = r\sin\theta$ とおくと，$E\colon 0 \leqq r \leqq \sqrt{\cos 2\theta}$, $-\dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{4}$ は D に写るので，

$$\iint_D \frac{dxdy}{(1+x^2+y^2)^2} = 2\int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr$$
$$= 2\int_0^{\frac{\pi}{4}} \left[-\frac{1}{2}\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta}{1+\cos 2\theta} d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1 - \tan^2\theta\right) d\theta = \frac{1}{2}\left(\frac{\pi}{4} - [\tan\theta - \theta]_0^{\frac{\pi}{4}}\right) = \frac{\pi - 2}{4}.$$

(10) $x = r\cos\theta$，$y = r\sin\theta$ とおくと，

$$\iint_D \left(|x|^3 + |y|^3\right) dxdy = \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} d\theta \int_0^5 r^4\left(|\cos^3\theta| + |\sin^3\theta|\right) dr = 625\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} \left(|\cos^3\theta| + |\sin^3\theta|\right) d\theta.$$

ここで，

$$\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} |\cos^3\theta| d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta d\theta - \int_{\frac{\pi}{2}}^{\frac{5}{4}\pi} \cos^3\theta d\theta = 2\int_0^{\frac{\pi}{2}} \cos^3\theta d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 3\theta + 3\cos\theta) d\theta = \frac{1}{2}\left[\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{4}{3}$$

であり，同様にして，$\displaystyle\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} |\sin^3\theta| d\theta = \frac{4}{3}$ なので，求める値は $\dfrac{5000}{3}$.

(11) $x = -3 + 3r\cos\theta$，$y = 3r\sin\theta$ と変換して，

$$\iint_D (x^2+y^2) dxdy = \int_0^{2\pi} d\theta \int_0^3 \left(r^3 - 3r^3\sin 2\theta + 9r\right) dr$$
$$= \int_0^{2\pi} \left(\frac{81}{4} - \frac{243}{4}\sin 2\theta + \frac{81}{2}\right) d\theta = \frac{243}{2}\pi.$$

(12) 対称性より，$x \geqq 0$，$y \geqq 0$ の場合を考えればよい．$x = u^2, y = v^2$ により，$E\colon u + v + \leqq 1, u \geqq 0, v \geqq 0$ は $\tilde{D}\colon \sqrt{x} + \sqrt{y} \leqq 1, x \geqq 0, y \geqq 0$ に写り，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2u & 0 \\ 0 & 2v \end{pmatrix} = 2uv$. よって，

$$
\begin{aligned}
&\iint_D |x|y^2 dxdy = 4\iint_{\tilde{D}} xy^2 dxdy = 16\int_0^1 dv \int_0^{1-v} u^3v^5 dv \\
&= 4\int_0^1 v^5(1-v)^4 dv = 4\int_0^1 \left(v^5 - 4v^6 + 6v^7 - 4v^8 + v^9\right) dv \\
&= 4\left[\frac{v^6}{6} - \frac{4}{7}v^6 + \frac{3}{4}v^8 - \frac{4}{9}v^9 + \frac{1}{10}v^{10}\right]_0^1 = \frac{1}{315}.
\end{aligned}
$$

(13) $D\colon \dfrac{24}{25}(x-1)^2 + \dfrac{36}{25}\left(y + \dfrac{1}{6}\right)^2 \leqq 1$ なので，$x = \dfrac{5}{2\sqrt{6}} r\cos\theta + 1$，$y = \dfrac{5}{6} r\sin\theta - \dfrac{1}{6}$ により，$E\colon 0 \leqq r \leqq 1, 0 \leqq \theta \leqq 2\pi$ は D に写る．また，

$$
\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{5}{2\sqrt{6}}\cos\theta & -\frac{5}{2\sqrt{6}} r\sin\theta \\ \frac{5}{6}\sin\theta & \frac{5}{6} r\cos\theta \end{pmatrix} = \frac{25}{12\sqrt{6}} r
$$

なので，

$$
\begin{aligned}
&\iint_D (x^2 + y^2) dxdy \\
&= \frac{25}{12\sqrt{6}}\int_0^{2\pi} d\theta \int_0^1 \left\{ \left(\frac{25}{24}\cos^2\theta + \frac{25}{36}\sin^2\theta\right) r^3 + \left(\frac{5}{\sqrt{6}}\cos\theta - \frac{5}{18}\cos\theta\right) r^2 + \frac{37}{36} r \right\} dr \\
&= \frac{25}{12\sqrt{6}}\int_0^{2\pi} \left\{ \frac{1}{4}\left(\frac{25}{24}\frac{1 + \cos 2\theta}{2} + \frac{25}{36}\frac{1 - \cos 2\theta}{2}\right) + \frac{1}{3}\left(\frac{5}{\sqrt{6}}\cos\theta - \frac{5}{18}\sin\theta\right) + \frac{37}{72} \right\} d\theta \\
&= \frac{25}{6\sqrt{6}}\pi\left(\frac{25}{192} + \frac{25}{288} + \frac{37}{72}\right) = \frac{10525}{3456\sqrt{6}}.
\end{aligned}
$$