1.7 確率変数の独立性と大数の法則

[1] (1)

X	0	1	2	計
$P(X=x)$	$1/2$	$1/3$	$1/6$	1

Y	0	1	2	計
$P(Y=y)$	$1/2$	$1/3$	$1/6$	1

(2) $P(X-Y=0)=P(X=Y=0)+P(X=Y=1)+P(X=Y=2)$

$$=\frac{1}{4}+\frac{1}{9}+\frac{1}{36}=\frac{14}{36}=\frac{7}{18}$$

(3) $P(X=0,Y=0)=\frac{1}{4}=\frac{1}{2}\times\frac{1}{2}=P(X=0)P(Y=0)$

$P(X=0,Y=1)=\frac{1}{6}=\frac{1}{2}\times\frac{1}{3}=P(X=0)\cdot P(Y=1)$

同様にして $P(X=i, Y=j)=P(X=i)\cdot P(Y=j)$ $(i=0,1,2,\ j=0,1,2)$ が成立. よって

XとYは独立

[2]
$$P(1/2\leq X<1,\ 0\leq Y\leq 1/2)=\iint_{\{(x,y):\,1/2\leq x<1,\ 0\leq y\leq 1/2\}} p(x,y)\,dx\,dy \xrightarrow{X\text{と}Y\text{は独立なので } p(x,y)=p_1(x)p_2(y)} \iint_{\{(x,y):\,1/2\leq x<1,\ 0\leq y\leq 1/2\}} p_1(x)\cdot p_2(y)\,dx\,dy$$

$$=\left(\int_{1/2}^{1}p_1(x)dx\right)\left(\int_{0}^{1/2}p_2(y)dy\right)=\left(\int_{1/2}^{1}2x\,dx\right)\cdot\left(\int_{0}^{1/2}2y\,dy\right)=\left[x^2\right]_{1/2}^{1}\cdot\left[y^2\right]_{0}^{1/2}=\frac{3}{4}\cdot\frac{1}{4}=\frac{3}{16}$$

[3]
$$P(|X-10|\leq 2)=1-P\left(|X-E(X)|>\frac{2}{\sqrt{3}}\sqrt{V(X)}\right)\geq 1-\left(\frac{\sqrt{3}}{2}\right)^2=1-\frac{3}{4}=\frac{1}{4}$$

[4]
$$E(X_i)=\int_{-1/2}^{1/2}x\cdot 1\,dx=0.\quad V(X_i)=\int_{-1/2}^{1/2}x^2\cdot 1\,dx=2\int_{0}^{1/2}x^2dx=\frac{2}{3}\left[x^3\right]_0^{1/2}=\frac{2}{3}\times\frac{1}{8}=\frac{1}{12}$$

$$\therefore E(\bar{X})=0,\quad V(\bar{X})=\frac{1}{n^2}\sum_{i=1}^{n}V(X_i)=\frac{1}{n^2}\cdot\frac{n}{12}=\frac{1}{12n}$$

$$\therefore P(|\bar{X}|\leq 1/10)=P\left(|\bar{X}-E(\bar{X})|\leq\frac{\sqrt{12n}}{10}\sqrt{V(\bar{X})}\right)$$

$$\fallingdotseq 1-P\left(|\bar{X}-E(\bar{X})|>\frac{\sqrt{12n}}{10}\sqrt{V(\bar{X})}\right)$$

$$\geq 1-\frac{100}{12n}\geq\frac{9}{10}$$

$$\therefore\frac{100}{12n}\leq\frac{1}{10}\quad\therefore 12n\geq 10\times 100\quad\therefore n\geq\frac{500}{6}\fallingdotseq 83.3\cdots\quad\therefore n\geq 84$$

$\therefore n=84$