1.6 同時確率分布

2 (1), (2):

X	0	1	計
$P(X=x)$	0.6	0.4	1

Y	0	1	2	計
$P(Y=y)$	0.5	0.3	0.2	1

(3) $E(X)=0\times0.6+1\times0.4=0.4$, $E(Y)=0\times0.5+1\times0.3+2\times0.2=0.3+0.4=0.7$

$$V(X)=E(X^2)-\{E(X)\}^2=0^2\times0.6+1^2\times0.4-0.4^2=0.4-0.16=0.24$$
$$V(Y)=E(Y^2)-\{E(Y)\}^2=0^2\times0.5+1^2\times0.3+2^2\times0.2-0.7^2=0.3+0.8-0.49=0.61$$
$$\gamma(X,Y)=E(X\cdot Y)-E(X)\cdot E(Y)=0\times0\times0.3+0\times1\times0.2+0\times2\times0.1+1\times0\times0.2+1\times1\times0.1$$
$$+1\times2\times0.1-0.4\times0.7=0.1+0.2-0.28=0.02$$

(4) $P(X+Y\leq1)=P(X=0,Y=0)+P(X=0,Y=1)+P(X=1,Y=0)=0.3+0.2+0.2=0.7$

3 (1), (2), (3): $P(X=1,Y=1)=\frac{2}{6}\times\frac{1}{2}=\frac{1}{6}$, $P(X=1,Y=2)=\frac{2}{6}\times\frac{1}{2}=\frac{1}{6}$,
$P(X=2,Y=1)=\frac{4}{6}\times\frac{1}{2}=\frac{1}{3}$, $P(X=2,Y=2)=\frac{4}{6}\times\frac{1}{2}=\frac{1}{3}$

$Y \backslash X$	1	2	Yの周辺確率
1	1/6	1/3	1/2
2	1/6	1/3	1/2
Xの周辺確率	1/3	2/3	1

(3) $E(X)=1\times\frac{1}{3}+2\times\frac{2}{3}=\frac{1}{3}+\frac{4}{3}=\frac{5}{3}$
$E(Y)=1\times\frac{1}{2}+2\times\frac{1}{2}=\frac{1}{2}+1=\frac{3}{2}$

$$V(X)=E(X^2)-\{E(X)\}^2=1^2\times\frac{1}{3}+2^2\times\frac{2}{3}-\left(\frac{5}{3}\right)^2=\frac{1}{3}+\frac{8}{3}-\frac{25}{9}=\frac{27}{9}-\frac{25}{9}=\frac{2}{9}$$
$$V(Y)=E(Y^2)-\{E(Y)\}^2=1^2\times\frac{1}{2}+2^2\times\frac{1}{2}-\left(\frac{3}{2}\right)^2=\frac{1}{2}+2-\frac{9}{4}=\frac{10}{4}-\frac{9}{4}=\frac{1}{4}$$
$$\gamma(X,Y)=E(X\cdot Y)-E(X)\cdot E(Y)=1\times1\times\frac{1}{6}+1\times2\times\frac{1}{6}+2\times1\times\frac{1}{3}+2\times2\times\frac{1}{3}-\frac{5}{3}\times\frac{3}{2}$$
$$=\frac{1}{6}+\frac{2}{6}+\frac{4}{6}+\frac{8}{6}-\frac{15}{6}=\frac{15}{6}-\frac{15}{6}=0$$

4 (1), (2), (3):
$P(X=1,Y=1)=\frac{2}{6}\times\frac{1}{5}=\frac{1}{15}$, $P(X=1,Y=2)=\frac{2}{6}\times\frac{4}{5}=\frac{4}{15}$, $P(X=2,Y=1)=\frac{4}{6}\times\frac{2}{5}=\frac{4}{15}$
$P(X=2,Y=2)=\frac{4}{6}\times\frac{3}{5}=\frac{6}{15}$

$Y \backslash X$	1	2	Yの周辺確率
1	1/15	4/15	1/3
2	4/15	6/15	2/3
Xの周辺確率	1/3	2/3	1

(4) $E(X)=1\times\frac{1}{3}+2\times\frac{2}{3}=\frac{5}{3}$
$E(Y)=1\times\frac{1}{3}+2\times\frac{2}{3}=\frac{5}{3}$

$$V(X)=E(X^2)-\{E(X)\}^2=1^2\times\frac{1}{3}+2^2\times\frac{2}{3}-\left(\frac{5}{3}\right)^2$$
$$=\frac{1}{3}+\frac{8}{3}-\frac{25}{9}=\frac{27}{9}-\frac{25}{9}=\frac{2}{9}$$

$V(Y)=E(Y^2)-\{E(Y)\}^2=1^2\times\frac{1}{3}+2^2\times\frac{2}{3}-\left(\frac{5}{3}\right)^2=\frac{1}{3}+\frac{8}{3}-\frac{25}{9}=\frac{2}{9}$

$\sigma(X,Y)=E(X\cdot Y)-E(X)\cdot E(Y)=1\times1\times\frac{1}{15}+1\times2\times\frac{4}{15}+2\times1\times\frac{4}{15}+2\times2\times\frac{6}{15}-\frac{5}{3}\times\frac{5}{3}$

$=\frac{1}{15}+\frac{8}{15}+\frac{8}{15}+\frac{24}{15}-\frac{25}{9}=\frac{41}{15}-\frac{25}{9}=\frac{246}{90}-\frac{250}{90}=-\frac{4}{90}=-\frac{2}{45}$

$\boxed{5}$ (1) ・$-\frac{1}{2}\leq x<\frac{1}{2}$ のとき: $p_1(x)=\int_{-1}^{1}\frac{1}{2}dy=1$

・それ以外のxのとき: $p_1(x)=\int_{-\infty}^{\infty}0\,dy=0$ ∴ $p_1(x)=\begin{cases}1 & (-\frac{1}{2}\leq x<\frac{1}{2})\\ 0 & (\text{それ以外})\end{cases}$

・$-1\leq y<1$ のとき: $p_2(y)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{2}dx=\frac{1}{2}$

・それ以外のyのとき: $p_2(y)=\int_{-\infty}^{\infty}0\,dx=0$ ∴ $p_2(y)=\begin{cases}\frac{1}{2} & (-1\leq y<1)\\ 0 & (\text{それ以外})\end{cases}$

(2) $P(X\geq0,Y\geq0)=\int_0^{\infty}\int_0^{\infty}f(x,y)dxdy=\int_0^{1/2}\int_0^{1}1/2\,dxdy=\left(\int_0^{1/2}1\,dx\right)\left(\int_0^{1}\frac{1}{2}dy\right)$

$=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

(3) $E(X)=\int_{-\frac{1}{2}}^{\frac{1}{2}}x\,dx=0$, $E(Y)=\int_{-1}^{1}\frac{1}{2}y\,dy=0$

$V(X)=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2dx=2\int_0^{1/2}x^2dx=\frac{2}{3}\left[x^3\right]_0^{1/2}=\frac{2}{3}\times\frac{1}{8}=\frac{1}{12}$

$V(Y)=\int_{-1}^{1}\frac{1}{2}y^2dy=\int_0^{1}y^2dy=\frac{1}{3}\left[y^3\right]_0^1=\frac{1}{3}$

$\sigma(X,Y)=E(X\cdot Y)=\int_{-1/2}^{1/2}\int_{-1}^{1}x\cdot y\cdot\frac{1}{2}dxdy=\frac{1}{2}\left(\int_{-1/2}^{1/2}x\,dx\right)\left(\int_{-1}^{1}y\,dy\right)=0$

$\boxed{6}$ (1) ・$x\geq0$ のとき: $p_1(x)=\int_0^{\infty}\frac{1}{2}e^{-x}\cdot e^{-\frac{y}{2}}dy=\frac{1}{2}e^{-x}\int_0^{\infty}e^{-\frac{y}{2}}dy$. ここで

$\int_0^{\infty}e^{-\frac{y}{2}}dy=\left[-2e^{-\frac{y}{2}}\right]_0^{\infty}=-2(0-1)=2$ ∴ $p_1(x)=e^{-x}$

・それ以外のxのとき: $p_1(x)=\int_{-\infty}^{\infty}0\,dy=0$ ∴ $p_1(x)=\begin{cases}e^{-x} & (x\geq0)\\ 0 & (x<0)\end{cases}$

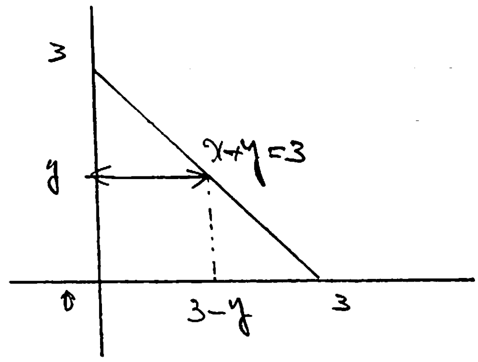
・$y\geq0$ のとき: $p_2(y)=\int_0^{\infty}\frac{1}{2}e^{-x}\cdot e^{-\frac{y}{2}}dx=\frac{1}{2}e^{-\frac{y}{2}}\int_0^{\infty}e^{-x}dx$. ここで $\int_0^{\infty}e^{-x}dx=\left[-e^{-x}\right]_0^{\infty}=1$

$\therefore\ f_2(y) = \frac{1}{2} e^{-\frac{y}{2}}$.

・その他の y のとき：$f_2(y) = \int_{-\infty}^{\infty} 0\, dx = 0$ $\quad \therefore\ f_2(y) = \begin{cases} \frac{1}{2} e^{-\frac{y}{2}} & (y \geqq 0) \\ 0 & (y < 0) \end{cases}$

(2) $P(X+Y \leqq 3) = \iint_{\{(x,y)\,:\, x+y \leqq 3\}} f(x,y)\,dx\,dy = \iint_D \frac{1}{2} e^{-x} \cdot e^{-\frac{y}{2}}\,dx\,dy = (*)$

但し $D := \{(x,y) : x \geqq 0,\ y \geqq 0,\ x+y \leqq 3\}$



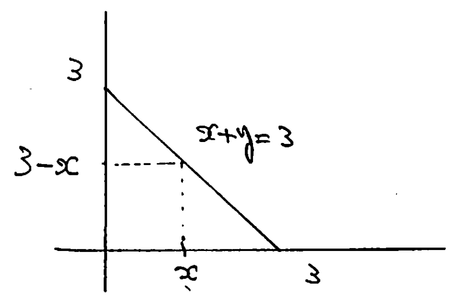
$(*) = \int_0^3 \frac{1}{2} e^{-\frac{y}{2}} dy \int_0^{3-y} e^{-x} dx$

ここで $\int_0^{3-y} e^{-x} dx = \left[-e^{-x}\right]_0^{3-y} = 1 - e^{y-3}$

$\therefore (*) = \int_0^3 \frac{1}{2} e^{-\frac{y}{2}} (1 - e^{y-3}) dy = \frac{1}{2}\int_0^3 e^{-\frac{y}{2}} dy - \frac{1}{2}\int_0^3 e^{\frac{y}{2}-3} dy$

$= \frac{1}{2}\left[-2e^{-\frac{y}{2}}\right]_0^3 - \frac{1}{2}\left[2e^{\frac{y}{2}-3}\right]_0^3 = \frac{1}{2}\cdot 2(1-e^{-\frac{3}{2}}) - \frac{1}{2}\cdot 2\,(e^{-\frac{3}{2}} - e^{-3})$

$= 1 - e^{-\frac{3}{2}} - e^{-\frac{3}{2}} + e^{-3} = 1 - 2e^{-\frac{3}{2}} + e^{-3}$



(*)の計算の別法

$(*) = \int_0^3 \frac{1}{2} e^{-x} dx \int_0^{3-x} e^{-\frac{y}{2}} dy$

ここで $\int_0^{3-x} e^{-\frac{y}{2}} dy = \left[-2e^{-\frac{y}{2}}\right]_0^{3-x} = -2\left(e^{-\frac{3}{2}+\frac{x}{2}} - 1\right) = 2 - 2e^{-\frac{3}{2}+\frac{x}{2}}$

$\therefore (*) = \int_0^3 \frac{1}{2} e^{-x}\left(2 - 2e^{-\frac{3}{2}+\frac{x}{2}}\right) dx = \int_0^3 \left(e^{-x} - e^{-\frac{3}{2}-\frac{x}{2}}\right) dx = \left[-e^{-x} + 2e^{-\frac{3}{2}-\frac{x}{2}}\right]_0^3$

$= \left(-e^{-3} + 2e^{-\frac{3}{2}-\frac{3}{2}}\right) - \left(-1 + 2e^{-\frac{3}{2}}\right) = -e^{-3} + 2e^{-3} + 1 - 2e^{-\frac{3}{2}} = 1 - 2e^{-\frac{3}{2}} + e^{-3}$