Staff Paper Series '10-01 May 2010

Voluntary Participation and Provision of Public Goods in Large Finite Economies

Hideo Konishi
Department of Economics, Boston College
and
Ryusuke Shinohara
Faculty of Economics, Shinshu University

Faculty of Economics
Shinshu University
Matsumoto 390-8621 Japan
Phone: +81-263-35-4600
Fax: +81-263-37-2344

Voluntary Participation and Provision of Public Goods in Large Finite Economies*

Hideo Konishi† Ryusuke Shinohara‡

May 19, 2010

Abstract

How serious is the free-riding problem when agents' participation in contribution to the public goods is voluntary? Is free-riding incentive affected by the number of agents? In a voluntary participation game, agents decide whether to participate in the provision or not. Agents who participate provide a public good and pay the fees according to an allocation rule. Agents who non-participate free-ride on the participants. We examine how the equilibrium public good provision level changes as the economy is replicated in the sense of Milleron (1972). We introduce a continuity concept for an allocation rule, the *uniform continuity in replication* (UCR), which is satisfied by many mechanisms. We show that if an allocation rule satisfies UCR, then the equilibrium level of the public good converges to zero as the economy is replicated.

Keywords. public good provision, participation game, replicated economy.

JEL Classification Numbers. C72, H41.

---

*We thank P.J. Healy, Koji Takamiya, Yukio Koriyama and Hervé Moulin for their helpful conversations. We also thank participants at Hitotsubashi COE Lectures and Workshop on Choice, Games, and Welfare (January, 2009), Public Economics Workshop at Nagoya Gakuin University (May, 2009), and Theory Workshop at Niigata University (June, 2009).

†Department of Economics, Boston College, 140 Commonwealth Avenue Chestnut Hill, MA 02467, USA. (E-mail) hideo.konishi@bc.edu

‡Faculty of Economics, Shinshu University, 3-1-1, Asahi, Matsumoto, Nagano 390-8621, Japan. (E-mail) ryusukes@shinshu-u.ac.jp

1 Introduction

How serious is the free-rider problem when agents' participation in contribution to the public good is voluntary? Does the number of agents in the economy matter? In this paper, we investigate how the equilibrium provision of a public good is affected as the population increases by adopting Milleron's (1972) notion of replication of a public goods economy.

It is well known that the provision of public goods is subject to free-riding incentives. Although Samuelson's (1954) view of this problem is pessimistic, Groves and Ledyard (1977) show that the efficient provision of public goods can be achieved in a Nash equilibrium by crafting a public good mechanism appropriately. Although the Groves-Ledyard mechanism does not satisfy individual rationality, Hurwicz (1979) and Walker (1981) show that the Lindahl rule is implementable: the Lindahl rule is individually rational as well as efficient. Subsequently, numerous further mechanisms have been proposed to satisfy additional desirable properties. They all assume, however, that agents must participate in the mechanism: i.e., they assume that agents have no freedom to leave the mechanism.

However, since the provision of pure public goods intrinsically involves the non-excludability of the benefits, assuming that all agents participate in the mechanism may not be appropriate if we allow agents to decide whether to participate in the provision of public goods or not.[1] Several studies have

[1]In a private good economy, individual rationality requirements and agents' participation incentives are the same, while in a public good economy, agents' participation

examined if agents are actually willing to participate in the public goods mechanism voluntarily. Saijo and Yamato (1999) consider a voluntary participation game in a divisible public goods case. They show a negative result on efficiency of public goods provision, and then characterize equilibrium participation of the mechanism that implements the Lindahl allocation rule in a symmetric Cobb-Douglas utility case. Subsequently, Saijo and Yamato (2010) extend their result to a more general domain, and show that for all agents the participation incentives in the mechanisms that implement the Lindahl rule become weaker as the number of agents in the economy increases.

Healy (2010) and Furusawa and Konishi (2009) also show that the non-participation (free-riding) problem becomes more serious as the population increases. Unlike Saijo and Yamato (2010), they examine how the equilibrium level of a public good changes as the population in the economy grows, following Milleron's (1972) definition of a replicated economy.[2] They prove that every Nash equilibrium level of a public good goes to zero as the economy is infinitely replicated under different conditions. Healy (2010) focuses on the *fixed contribution* rule with *equilibrium participation*. The fixed contribution rule assigns the same fee for each agent independent of the other

incentives depends on how many others participate.

[2]Muench (1972), Milleron (1972), and Conley (1994) discuss the difficulty of replicating a public goods economy and offer various possible methods. Milleron's notion of replication is to split endowments with replicates and adjust preferences so that agents' concerns for the private good are relative to the size of their endowments. This notion is employed by Healy (2010).

agents' participation. The equilibrium participation rule is such that under this rule, all agents voluntarily participate in the public goods provision.[3] He shows that fixed contribution and equilibrium participation imply no public goods provision in the limit on a domain of monotone and continuous preferences. Furusawa and Konishi (2009) use the *efficient allocation rule*, which associates an efficient allocation with respect to participants' preferences with each set of participants. They show that under efficient allocation rules, the public goods provision level converges to zero as population grows to infinity in the quasi-linear preference domain. Unlike Healy, their convergence result follows from the fact that the fraction of agents who contribute to public good provision converges to zero.

In this paper, we provide a sufficient condition under which the equilibrium level of a public good converges to zero as the economy is replicated in Milleron (1972)'s sense. We work on quasi-linear economies with divisible public goods. We introduce one condition on rules of public goods provision: **uniform continuity in replication (UCR)**. Considering replications of an economy, UCR requires that if the sets of participants are close in the composition of their population (in a particular sense), the public goods

[3] Introducing outside opportunities by a "reversion function" (each outcome is mapped to another outcome in the case of no participation), Jackson and Palfrey (2001) analyze the implementation problem including participation of all players when players' participation in a mechanism is voluntary. They extend the Maskin monotonicity condition to accommodate the voluntary participation problem. Although their reversion function is very general, it assigns the same outcome no matter who deviates from the original outcome. Thus, the method may not be suitable for a public goods provision problem in which different players' deviations from participation may generate different outcomes.

provision levels are *uniformly* close in all replicated economies. We impose no condition on cost-sharing of public goods provision, and we allow for budget-surplus. Although Milleron's replication does not affect the feasibility of aggregated allocations, UCR does not require that the public goods provision level is unaffected by replications. We show that if a public goods provision rule satisfies UCR, then all equilibrium public goods provision levels converge to zero as an economy is replicated (Theorem 1). The class of rules that satisfy UCR contains many popular allocation rules in mechanism design theory. The efficient allocation rules such as the Lindahl and the core allocation rules satisfy UCR, and so does the Clarke (1971) rule. All fixed contribution rules with or without equilibrium participation also satisfy UCR on the quasi-linear preference domain. Hence, using our result, we can show that, under the mechanisms that are popular in mechanism design theory, the equilibrium level of the public good diminishes to zero as the economy is replicated at least in the domain of quasi-linear preferences.

Our result also shows that if some positive public good is provided at a Nash equilibrium in the limit, an allocation rule has discontinuity around the Nash equilibrium level of the public good. The discontinuous allocation rule has a structure similar to the model of the provision of a discrete public good such as Palfrey and Rosenthal (1984) and Shinohara (2009).

This paper is organized as follows: in Section 2, we introduce the participation game with a public good, and in Section 3, we introduce UCR and present the main result. Section 4 provides examples of public goods mechanisms that satisfies UCR. Section 5 concludes the paper.

2 The Model

There are one private good and one public good in the economy. The preference domain we consider is quasi-linear in the private good, i.e., each agent's preference is represented by a utility function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that $U(z, g) = z + u(g)$, where z and g are private and public goods consumption levels, respectively, and $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, strictly increasing and concave function with $u(0) = 0$, $0 < \lim_{g \to 0} u'(g) < M$ (bounded willingness-to-pay). We can weaken the bounded willingness-to-pay condition, but at the cost of more involved proofs. Let $\mathcal{U}$ be the domain of all preferences that satisfy the above properties (the bound $M > 0$ is common to all players). Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable, strictly increasing and convex cost function of public goods provision such that $C(0) = 0$, $\lim_{g \to 0} C'(g) = \underline{c} > 0$, and for all $c > \underline{c}$ there is $g > 0$ with $C'(g) > c$. The last condition requires that the marginal cost of production is bounded below by a positive value $\underline{c}$. Let $\mathcal{C}$ be the collection of cost functions that satisfy the above properties. Let $\nu : \mathcal{U} \to \mathbb{Z}_+$ be a **population allocation** that describes the population distribution of preference types. For all $\nu \in \mathcal{N}$, let $Supp(\nu) \equiv \{u \in \mathcal{U} | \nu(u) > 0\}$ and let $|Supp(\nu)|$ be the cardinality of $Supp(\nu)$. Let $\mathcal{N}$ be the collection of all (finite) population allocations that satisfy $|Supp(\nu)| < \infty$. For all $\nu, \nu' \in \mathcal{N}$, $\nu' \leq \nu$ if and only if $\nu'(u) \leq \nu(u)$ for all $u \in \mathcal{U}$. For all $\nu \in \mathcal{N}$, $|\nu| \equiv \sum_{u \in \mathcal{U}} \nu(u)$. For all $\nu, \nu' \in \mathcal{N}$, $|\nu - \nu'| \equiv \sum_{u \in \mathcal{U}} |\nu(u) - \nu'(u)|$.

An **economy** is a pair (ν, C), and $\mathcal{E}$ denotes the collection of all economies. Below, we define allocation rules which map an economy (ν, C) and its subpopulation $\nu' \leq \nu$ (participants of the mechanism) to public goods provision level g and the cost-sharing among participants τ. An (anonymous) **transfer function** $\tau : \mathcal{U} \to \mathbb{R}$ assigns a payment amount to each type of agent, and $\mathcal{T}$ denotes the collection of all transfer functions. An **allocation rule** is a function $\varphi : \mathcal{N} \times \mathcal{N} \times \mathcal{C} \to \mathcal{T} \times \mathbb{R}_+$ such that (i) $\varphi_{\mathcal{T}}[\nu'; \nu, C] : \mathcal{U} \to \mathbb{R}$ satisfies $\varphi_{\mathcal{T}}[\nu'; \nu, C](u) = 0$ for all $u \notin Supp(\nu')$, and (ii)

$$\sum_{u \in \mathcal{U}} \nu'(u) \times \varphi_{\mathcal{T}}[\nu'; \nu, C](u) \geq C(\varphi_G[\nu'; \nu, C]),$$

where $\varphi_{\mathcal{T}}[\nu'; \nu, C] : \mathcal{U} \to \mathbb{R}$ is a transfer function that assigns a transfer payment to each existing type of agent $u \in \mathcal{U}$ with $\nu'(u) > 0$ in economy (ν, C), and $\varphi_G[\nu'; \nu, C] \in \mathbb{R}_+$ assigns the amount of public goods. Condition (ii) requires that the public goods provision level is feasible.

In the economy (ν, C), there exist $\nu(u_i)$ agents whose valuation function is u_i. Denote a generic u_i-type agent as $i(q)$ for $q \in \{1, \ldots, \nu(u_i)\}$. The set of agents in the economy is $N^\nu \equiv \{i(q) : u_i \in Supp(\nu) \text{ and } q \in \{1, ..., \nu(u_i)\}\}$. A **participation game** in $(\nu, C) \in \mathcal{E}$ with φ, $\Gamma(\nu, C, \varphi)$, is a list $(N^\nu, (\{0, 1\}, h_{i(q)})_{i(q) \in N^\nu})$, where $\{0, 1\}$ is a common strategy set and $h_{i(q)}$ is the player $i(q)$'s payoff function. Each agent $i(q)$ chooses 1 (participation) or 0 (non-participation), simultaneously. Let $s \in \{0,1\}^{|N^\nu|}$ be a profile of the decisions. Then, the set of participants is determined. Let $\mu^s : \mathcal{U} \to \mathbb{Z}_+$ be such that $\mu^s(u_i) \equiv |\{q| \, s_{i(q)} = 1\}|$ for all $u_i \in Supp(\nu)$, and $\mu^s(u_j) = 0$ for all $u_j \notin Supp(\nu)$. Note that μ^s is the population distribution of participants at s and it is a population distribution function. The payoff of $i(q)$ at s is

$$h_{i(q)}(s_{i(q)}, s_{-i(q)}) = \begin{cases} u_i(\varphi_G[\mu^s;\nu,C]) - \varphi_{\mathcal{T}}[\mu^s;\nu,C](u_i) & \text{if } s_{i(q)} = 1 \\ u_i(\varphi_G[\mu^s;\nu,C]) & \text{if } s_{i(q)} = 0 \end{cases}.$$

A **Nash equilibrium** s of a participation game $\Gamma(\nu, C, \varphi)$ is such that for all $i(q) \in N^\nu$, $h_{i(q)}(s_{i(q)}, s_{-i(q)}) \geq h_{i(q)}(s'_{i(q)}, s_{-i(q)})$ holds, where $s'_{i(q)} \neq s_{i(q)}$. We denote the set of Nash equilibria by $NE(\Gamma(\nu, C, \varphi))$. Without any confusion, we abuse some notation: we write $\varphi_G(m; n, C) = \varphi_G[\mu; \nu, C]$, where $m, n \in \mathbb{Z}_+^{|Supp(\nu)|}$ is such that $m \equiv (\mu(u_j))_{u_j \in Supp(\nu)}$ and $n \equiv (\nu(u_j))_{u_j \in Supp(\nu)}$. Similarly, let $\varphi_i(m; n, C) = \varphi_{\mathcal{T}}[\mu; \nu, C](u_i)$ for $u_i \in Supp(\nu)$. We define $m(s) \equiv (\mu^s(u))_{u \in Supp(\nu)}$ for all $s \in \{0,1\}^{|N^\nu|}$. For all $s \in \{0,1\}^{|N^\nu|}$ and all i such that $u_i \in Supp(\nu)$, $m_{-i}(s) \equiv (m_j(s))_{j \neq i}$. With these defined, we can rewrite the definition of Nash equilibrium: a strategy profile s is a Nash equilibrium, if and only if

(i) for all $u_i \in Supp(\nu)$ with $m_i(s) \geq 1$,

$$u_i(\varphi_G(m(s); n, C)) - \varphi_i(m(s); n, C) \geq u_i(\varphi_G(m_i(s) - 1, m_{-i}(s); n, C)), \text{ and}$$

(ii) for all $u_i \in Supp(\nu)$ with $m_i(s) \leq n_i - 1$,

$$u_i(\varphi_G(m(s); n, C)) \geq u_i(\varphi_G(m_i(s)+1, m_{-i}(s); n, C)) - \varphi_i(m_i(s)+1, m_{-i}(s); n, C).$$

Thus, by using (i), we have the following lemma, which will be useful in obtaining our main result in the next section.

Lemma 1 *A necessary condition for a strategy profile to be a Nash equilibrium is*

$$\begin{aligned} & \sum_{u_i \in Supp(\nu)} m_i(s) \left[\varphi_G(m(s); n, C) - \varphi_G(m_i(s) - 1, m_{-i}(s); n, C) \right] \\ \geq & \quad \frac{C(\varphi_G(m(s); n, C))}{M}. \end{aligned}$$

Proof. By using condition (i) of the necessary and sufficient condition for Nash equilibrium, we have

$$\begin{aligned} & \sum_{u_i \in Supp(\nu)} m_i(s) \times u_i(\varphi_G(m(s); n, C)) - \sum_{u_i \in Supp(\nu)} m_i(s) \varphi_i(m(s); n, C) \\ \geq & \sum_{u_i \in Supp(\nu)} m_i(s) u_i(\varphi_G(m_i(s) - 1, m_{-i}(s); n, C)), \end{aligned}$$

or

$$\begin{aligned}
&\sum_{u_i \in Supp(\nu)} m_i(s) \times u_i(\varphi_G(m(s); n, C)) \\
&\qquad - \sum_{u_i \in Supp(\nu)} m_i(s) u_i(\varphi_G(m_i(s) - 1, m_{-i}(s); n, C)) \\
\geq &\sum_{u_i \in Supp(\nu)} m_i(s) \varphi_i(m(s); n, C) \\
\geq &\; C(\varphi_G(m(s); n, C)).
\end{aligned}$$

The last inequality holds by the feasibility constraint.

Since $\lim_{g \to 0} u_i'(g) < M$ for all i, we have

$$\begin{aligned}
&\sum_{u_i \in Supp(\nu)} m_i(s) \left[u_i(\varphi_G(m(s); n, C)) - u_i(\varphi_G(m_i(s) - 1, m_{-i}(s); n, C)) \right] \\
\leq &\; M \times \sum_{u_i \in Supp(\nu)} m_i(s) \left[\varphi_G(m(s); n, C) - \varphi_G(m_i(s) - 1, m_{-i}(s); n, C) \right].
\end{aligned}$$

Thus, the following condition is necessary for s to be a Nash equilibrium:

$$\begin{aligned}
&\sum_{u_i \in Supp(\nu)} m_i(s) \left[\varphi_G(m(s); n, C) - \varphi_G(m_i(s) - 1, m_{-i}(s); n, C) \right] \\
\geq &\; \frac{C(\varphi_G(m(s); n, C))}{M}.
\end{aligned}$$

■

3 Uniform Continuity in Replications

This section introduces our main condition, uniform continuity in replications, and proves the main result of this paper. First, we define a replica of our economy. For all $u \in \mathcal{U}$ and $r \in \mathbb{Z}_{++}$, let $u^r \in \mathcal{U}$ be such that $u^r(g) = \frac{1}{r}u(g)$ for all $g \in \mathbb{R}_+$. For all $(\nu, C) \in \mathcal{E}$, let $(\nu^r, C) \in \mathcal{E}$ be **r-replication** of (ν, C) such that $\nu^r(u^r) = r\nu(u)$ for all $u \in \mathcal{U}$. This way of replicating a public goods economy is first defined by Milleron (1972).[4] It basically divides each agent into smaller pieces as replication proceeds. One of the merits of Milleron's notion of replication is that the efficient level of public goods provision is invariant with replications. Note, however, that $Supp(v) \neq Supp(v^r)$ for all $r > 1$, since preferences are altered after replication. We impose the following condition only on public goods provision levels (not on transfer functions).

Condition 1 (UCR — uniform continuity in replications): For all $e = (\nu, C) \in \mathcal{E}$, and all $\epsilon > 0$, there exists $\delta > 0$ such that for all $r \in \mathbb{Z}_{++}$ and all $\widetilde{\nu}^r, \bar{\nu}^r \in \mathcal{N}$ such that $\widetilde{\nu}^r \leq \bar{\nu}^r \leq \nu^r$ and $\frac{|\widetilde{\nu}^r - \bar{\nu}^r|}{|\nu^r|} \leq \delta$,

$$|\varphi_G[\widetilde{\nu}^r; \nu^r, C] - \varphi_G[\bar{\nu}^r; \nu^r, C]| < \epsilon$$

holds.

[4]This simple definition is an adaptation from Milleron (1972), but in quasi-linear economies.

Here is an interpretation of UCR. It requires continuity of public goods provision in a population *uniformly* for all numbers of replications of economy. Let $e \in \mathcal{E}$. For all $\widetilde{\nu}$, $\bar{\nu} \in \mathcal{N}$ such that $\widetilde{\nu}$, $\bar{\nu} \leq \nu$, $\frac{|\widetilde{\nu}-\bar{\nu}|}{|\nu|}$ represents the ratio of the difference in populations between $\widetilde{\nu}$ and $\bar{\nu}$ based on the whole economy. For all two subpopulations, if the ratio is sufficiently small, then the levels of the public good at the subpopulations are close. Roughly speaking, UCR imposes that if the population compositions are close, then the levels of the public good are also close.

First, note that this condition *does not* require that the public goods provision level stays intact for all replications. That is, we *do not* demand $\varphi_G[\nu'; \nu, C] = \varphi_G[\nu^{r\prime}; \nu^r, C]$ for $r = 2, 3, ...$, where $\nu^{r\prime}$ is the r-replication of ν'. Second, requiring "uniform" continuity is important: if "there exists $\delta > 0$ such that for all $r \in \mathbb{Z}_{++}$ and all ..." is replaced by "for all $r \in \mathbb{Z}_{++}$ there exists $\delta > 0$ such that for all ..." then the statement of UCR does not impose any restriction. This is because for all $\epsilon > 0$, we can pick $\delta > 0$ small enough to let $\frac{|\widetilde{\nu}^r-\bar{\nu}^r|}{|\nu^r|} \leq \delta$ imply $\widetilde{\nu}^r = \bar{\nu}^r$, which automatically guarantees $\varphi_G[\widetilde{\nu}^r; \nu^r, C] = \varphi_G[\bar{\nu}^r; \nu^r, C]$. Third, φ_G is *not* required to be continuous in similarity of preferences (for example, in Hausdorff distance). Our continuity is only in population composition, so we can allow rules that assign very different public goods provision levels when consumers' preferences change slightly. Fourth, this condition is completely silent about cost sharing of public goods. This somewhat innocuous-looking technical condition plays a central role in the subsequent analysis.

We now redefine a few notions for a replicated economy. Let $r \in \mathbb{Z}_{++}$ and let $(\nu^r, C) \in \mathcal{E}$. Let $\varphi_G(m^r; n^r, C) \equiv \varphi_G[\mu^r; \nu^r, C]$ and $\varphi_i(m^r; n^r, C) \equiv \varphi_{\mathcal{T}}[\mu^r; \nu^r, C](u_i^r)$, in which $m^r \equiv (\mu^r(u_i^r))_{u_i^r \in Supp(\nu^r)}$ and $n^r \equiv (\nu^r(u_i^r))_{u_i^r \in Supp(\nu^r)}$. There are $r\nu(u_i)$ agents whose valuation functions are u_i^r. Denote a generic u_i^r-type agent as $i^r(q)$ for $q \in \{1, \ldots, r\nu(u_i)\}$. For all $s^r \in \{0,1\}^{|N^{\nu^r}|}$ and for all i^r such that $u_i^r \in Supp(\nu^r)$, $m_{i^r}(s^r) \equiv |\{q | s_{i^r(q)} = 1\}|$. The set of Nash equilibria in $\Gamma(\nu^r, C, \varphi)$ is denoted by $NE(\Gamma(\nu^r, C, \varphi))$. The following is the key result which will lead us to our main theorem.

Lemma 2 *Let $(\nu, C) \in \mathcal{E}$ denote an economy. Suppose that φ satisfies UCR. Then, for any positive amount of public good $g > 0$, there is a $\bar{r} \in \mathbb{R}_{++}$ such that, for any $r \geq \bar{r}$, no Nash equilibrium of $\Gamma(\nu^r, C, \varphi)$ exists in which g or more units of the public good are provided.*

Proof. Suppose, on the contrary, that there is some positive public good provision level $g > 0$ such that for all $\bar{r} > 0$, there exist $r \geq \bar{r}$ and a Nash equilibrium s^r providing g or more units of the public good. From this, we can construct a (sub)sequence of Nash equilibria $\{s^r\}$ such that $\varphi_G(m(s^r); n^r, C) \geq g$ in each s^r of this sequence.

First note that a replicated economy (ν^r, C) satisfies that $u_i^r(0) = 0$, and $\lim_{g \to 0} u_i^{r\prime}(g) < \frac{M}{r}$ (bounded willingness-to-pay) for all $u_i^r \in \mathcal{U}$ with $\nu^r(u_i^r) > 0$. This is because all $u_i^r \in \mathcal{U}$ with $\nu^r(u_i^r) > 0$ correspond to $u_i \in \mathcal{U}$ with $\nu(u_i) > 0$ with $u_i^r(g) = \frac{1}{r} u_i(g)$ for all $g \in \mathbb{R}_+$. Thus, each Nash equilibrium s^r of $\Gamma(\nu^r, C, \varphi)$ must satisfy

$$\sum_{u_i^r \in Supp(\nu^r)} m_{i^r}(s^r) \left[\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C)\right] \\ \geq \frac{C(\varphi_G(m(s^r); n^r, C))}{\frac{M}{r}}, \quad (1)$$

where, for any $u_i^r \in \mathcal{U}$, $m_{i^r}(s^r)$ is the number of u_i^r-type agents that choose participation at s^r and $m_{-i^r}(s^r) \equiv (m_{j^r}(s^r))_{u_j^r \neq u_i^r}$. Dividing both sides of (1) by $|\nu^r| = r|\nu|$, we have

$$\sum_{u_i^r \in Supp(\nu^r)} \frac{m_{i^r}(s^r)}{r|\nu|} \left[\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C)\right] \\ \geq \frac{C(\varphi_G(m(s^r); n^r, C))}{M|\nu|}. \quad (2)$$

In the following, we will show that the left-hand side of (2) diminishes to zero as $r \to \infty$. This, together with $\frac{C(\varphi_G(m(s^r);n^r,C))}{M|\nu|} > 0$, implies that (2) is violated for large r, and we can generate a contradiction.

We show that $\varphi_G(m_{i^r}(s^r)-1, m_{-i^r}(s^r); n^r, C)$ converges to $\varphi_G(m(s^r); n^r, C)$ as $r \to \infty$.

Claim 1 *Let $\{s^r\}$ be a Nash-equilibrium (sub)sequence with $\varphi_G(m(s^r); n^r, C) \geq g > 0$ for all r on the sequence. Then, for any $\epsilon > 0$, there is $r_\epsilon \in \mathbb{R}_{++}$ such that $\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C) < \epsilon$ for any $r \geq r_\epsilon$ on the sequence.*

Proof of Claim 1. Let $(\widetilde{\nu}^r(u_j^r))_{u_j^r \in Supp(\nu^r)} = m(s^r)$. Let $(\bar{\nu}^r(u_j^r))_{u_j^r \in Supp(\nu^r)}$ be such that $\bar{\nu}^r(u_{i^r}) = m_{i^r}(s^r) - 1$ for some u_i^r with $m_{i^r}(s^r) > 0$ and $\bar{\nu}^r(u_j^r) = m_{j^r}(s^r)$ for all $u_j^r \neq u_i^r$. Then, $\frac{|\bar{\nu}^r - \widetilde{\nu}^r|}{|\nu^r|} = \frac{1}{r|\nu|}$. Hence, there exists $r_\epsilon > 0$ such that $\frac{1}{r|\nu|} \leq \delta$ for all $r \geq r_\epsilon$, which implies that, for any $\epsilon > 0$, there is $r_\epsilon > 0$ such that $|\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C)| < \epsilon$ for any $r \geq r_\epsilon$ by condition 1 (UCR). Since $m(s^r)$ is a Nash equilibrium participation, the content of $|\cdot|$ is positive. **(End of Proof of Claim 1)**

Since $r|\nu| \geq m_{i^r}(s^r)$ for any u_i^r with $m_{i^r}(s^r) > 0$, we have

$$\sum_{u_i^r \in Supp(\nu^r)} \frac{m_{i^r}(s^r)}{r|\nu|} \left[\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C)\right]$$
$$\leq \sum_{u_i^r \in Supp(\nu^r)} \left[\varphi_G(m(s^r); n^r, C) - \varphi_G(m_{i^r}(s^r) - 1, m_{-i^r}(s^r); n^r, C)\right]. \quad (3)$$

From Claim 1 and $|Supp(\nu^r)| = |Supp(\nu)| < \infty$, the right-hand side of (3) diminishes to zero for all u_i^r as r goes to infinity. However, we have

$$\frac{C(\varphi_G(m(s^r), C))}{M|\nu|} \geq \frac{C(g)}{M|\nu|} > 0,$$

which indicates that s^r cannot be a Nash equilibrium of $\Gamma(\nu^r, C, \varphi)$ for sufficiently large r unless $g = 0$ by (2), since $|\nu|$ is the population of the original economy and is fixed. This is a contradiction. ■

Theorem 1 *For all economy $(\nu, C) \in \mathcal{E}$ with allocation rule φ, all Nash* *equilibrium public goods provision levels converge to zero as economy* (ν, C) *is replicated (i.e.,* $\lim_{r\to\infty} \varphi_G(m(s^r); n^r, C) = 0$ *for all Nash equilibrium sequence* $\{s^r\}_{r=1}^{\infty}$ *such that* $s^r \in NE(\Gamma(\nu^r, C, \varphi))$ *for all* $r = 1, 2, ...$*), if* φ_G *satisfies UCR.*

Proof. Suppose not: for some (ν, C), there exist $\bar{g} > 0$ and a (sub)sequence of Nash equilibria, $\{s^r\}_{r=1}^{\infty}$, such that, for all $r = 1, 2, \ldots$, $s^r \in NE(\Gamma(\nu^r, C, \varphi))$ and $\varphi_G(m(s^r); n^r, C) \geq \bar{g}$. However, it is clear from Lemma 2 that Nash-equilibrium sequences can not be constructed in such a way that s^r supports the provision of not less than $\bar{g}$ units of the public good for any $r = 1, 2, \ldots$. This is a contradiction. ■

Note that if condition UCR is guaranteed around the equilibrium provision of a public good, the convergence result can be established. For example, consider a rule which is continuous around the equilibrium levels but discontinuous in very high provision levels where the individual rationality condition is not satisfied for anybody; clearly, such a provision level cannot be achieved as a Nash equilibrium. Although this rule has discontinuity, all sequences converge to zero under this rule. Therefore, UCR is not a necessary condition for the convergence.

Conversely, the discontinuity around a Nash equilibrium may allow that a positive level of a public good is provided at an equilibrium even if the economy is replicated. In the conclusion, we will provide a simple example of this.

4 Examples of UCR Allocation Rules

We present allocation rules that satisfy UCR. A voluntary participation game with a perfectly divisible public good has been studied by several authors such as Saijo and Yamato (1999, 2010), Shinohara (2007), Furusawa and Konishi (2009), and Healy (2010). As we will see below, many allocation rules, which have been studied in these papers, satisfy UCR. Hence, every Nash equilibrium level of public goods converges to zero as the number of replications gets large. We start with efficient provision rule of a public good.

Efficient provision rules of a public good with budget feasibility. An *efficient provision rule of a public good with budget feasibility*, which is denoted by $\varphi^E = (\varphi^E_G, \varphi^E_T)$, is defined as follows: for all (ν, C) and all $\widetilde{\nu} \leq \nu$,

$$\varphi^E_G[\widetilde{\nu}; \nu, C] = \arg\max_{g \geq 0} \sum_{u \in Supp(\widetilde{\nu})} \widetilde{\nu}(u)\, u(g) - C(g), \text{ and}$$

$$\sum_{u \in Supp(\widetilde{\nu})} \widetilde{\nu}(u)\, \varphi^E_T[\widetilde{\nu}; \nu, C](u) \geq C(\varphi^E_G[\widetilde{\nu}; \nu, C]).$$

Rule φ^E_G assigns a public good of a level that maximizes the surplus of participants and the sum of the payments from the participants covers its production cost. Ifφ^E satisfies the budget feasibility condition with equality, then it is called an *efficient allocation rule.* The Lindahl (ratio) and the core allocation rules are such examples.

A strategy-proof mechanism is also in this class. The Clarke mechanism, introduced by Clarke (1971), is such an example. The outcome obtained at dominant strategy equilibrium of the Clarke mechanism is representable by $\varphi^{Clarke} = (\varphi_G^{Clarke}, \varphi_{\mathcal{T}}^{Clarke})$ in the following way: for all (ν, C) and for all $\widetilde{\nu} \leq \nu$, $\varphi_G^{Clarke}[\widetilde{\nu}; \nu, C] = \varphi_G^E[\widetilde{\nu}; \nu, C]$ and for all $u_i \in Supp(\widetilde{\nu})$,

$$\begin{aligned}\varphi_{\mathcal{T}}^{Clarke}[\widetilde{\nu}; \nu, C](u_i) =& \frac{C(\varphi_G^{Clarke}[\widetilde{\nu}; \nu, C])}{|\widetilde{\nu}|} \\ &+ \max_{g \geq 0} \left(\sum_{u \in \mathcal{U}} \widetilde{\nu}_{-i}(u) u(g) - \frac{(|\widetilde{\nu}| - 1) C(g)}{|\widetilde{\nu}|} \right) \\ &- \left(\sum_{u \in \mathcal{U}} \widetilde{\nu}_{-i}(u) u(\varphi_G^E[\widetilde{\nu}; \nu, C]) - \frac{(|\widetilde{\nu}| - 1) C(\varphi_G^E[\widetilde{\nu}; \nu, C])}{|\widetilde{\nu}|} \right),\end{aligned}$$

where $\widetilde{\nu}_{-i}$ is such that $\widetilde{\nu}_{-i}(u_i) = \widetilde{\nu}(u_i) - 1$ and $\widetilde{\nu}_{-i}(u) = \widetilde{\nu}(u)$ for all $u \neq u_i$. The Clarke rule satisfies budget feasibility. The class of Groves mechanisms contains the Clarke mechanism. Not all Groves mechanisms are budget feasible. If a Groves mechanism satisfies budget feasibility, then it is an efficient provision rule with budget feasibility.

In the following, we prove that φ_G^E satisfies UCR.

Proposition 1 *Rule φ_G^E satisfies UCR.*

Proof. Fix $\epsilon > 0$, $r \in \mathbb{Z}_{++}$, and $e^r = (\nu^r, C)$. For $\delta > 0$, let $k(r; \delta, \nu) = \delta r\, |\nu|$ and let $\underline{k(r; \delta, \nu)}$ be the largest integer k such that $k \leq k(r; \delta, \nu)$. Note that by assumption, there is $0 < M < \infty$ such that $M > u'(g)$ for all $u \in \mathcal{U}$. By the Samuelson rule, φ_G^E satisfies $\sum_{u \in \mathcal{U}} \hat{\nu}(u) u'(\varphi_G^E[\hat{\nu}; \nu, C]) = C'(\varphi_G^E[\hat{\nu}; \nu, C])$ for all $e = (\nu, C) \in \mathcal{E}$ and all $\hat{\nu} \leq \nu$. Let $\underline{\varphi_G}[\hat{\nu}, k; \nu, C]$ be the lowerbound for the public good provisions level when k consumers leave from economy

$(\hat{\nu}, k)$: i.e., $\underline{\varphi_G}[\hat{\nu}, k; \nu, C]$ is such that

$$\sum_{u \in \mathcal{U}} \hat{\nu}(u) u' \left(\underline{\varphi_G}[\hat{\nu}, k; \nu, C] \right) - kM = C' \left(\underline{\varphi_G}[\hat{\nu}, k; \nu, C] \right),$$

if there is a solution to the above equation, and $\underline{\varphi_G}[\hat{\nu}, k; \nu, C] = 0$, otherwise. Recall that u and C are continuously differentiable, u' is non-increasing, and C' is increasing in g. Thus, if there is a solution in the above equation, it must be unique.

Let $\overline{\Delta G(k; \nu, C)} \equiv \max_{\hat{\nu} \leq \nu} \left| \varphi_G^E[\hat{\nu}; \nu, C] - \underline{\varphi_G}[\hat{\nu}, k; \nu, C] \right|$: i.e., the maximum possible reduction of the public goods provision level in all possible public goods provision groups $\hat{\nu}$ of (ν, C) by k people leaving from the group $\hat{\nu}$. To go step further, let us extend this notion to a possibly continuous $\hat{\nu}$ case. We now allow for any (nonnegative) real-valued function $\hat{\nu} : \mathcal{U} \to \mathbb{R}_+$ such that $\hat{\nu}(u) \leq \nu(u)$ for all $u \in \mathcal{U}$. Since $\varphi_G^E[\hat{\nu}; \nu, C]$ and $\underline{\varphi_G}[\hat{\nu}, k; \nu, C]$ are continuous in $\hat{\nu}$ and $\hat{\nu}$ satisfies $0 \leq \hat{\nu} \leq \nu$, $\underline{\varphi_G}[\hat{\nu}, k; v, C]$ and $\overline{\Delta G(k; \nu, C)}$ are still well defined.

Now, we apply the above concept to replica economies of (ν, C). For all $\widetilde{\nu}^r, \bar{\nu}^r \in \mathcal{N}$ such that $\widetilde{\nu}^r \leq \bar{\nu}^r \leq \nu^r$ with $|\bar{\nu}^r - \widetilde{\nu}^r| \leq \underline{k(r; \delta, \nu)}$, we have

$$\begin{aligned} \left| \varphi_G^E[\bar{\nu}^r; \nu^r, C] - \varphi_G^E[\widetilde{\nu}^r; \nu^r, C] \right| &\leq \overline{\Delta G(\underline{k(r; \delta, \nu)}; \nu^r, C)} \\ &\leq \overline{\Delta G(k(r; \delta, \nu); \nu^r, C)} \\ &= \overline{\Delta G(\delta r \left| \nu \right|; \nu^r, C)} \\ &= \overline{\Delta G(\delta \left| \nu \right|; \nu, C)}. \end{aligned}$$

The equalities hold by the definitions of $k(r;\delta,\nu)$ and the Milleron replication, respectively. Let $\delta(\epsilon)$ be defined by $\epsilon = \overline{\Delta G(\delta(\epsilon)\,|\nu|\,;\nu,C)}$. Clearly, $\overline{\Delta G(\delta\,|\nu|\,;\nu,C)}$ is continuous and monotonically increasing in δ, and takes zero value at $\delta = 0$. Thus, $\delta(\epsilon) > 0$ holds for all $\epsilon > 0$. Take $\bar{\delta}(\epsilon)$ such that $0 < \bar{\delta}(\epsilon) < \delta(\epsilon)$. Then, φ_G^E satisfies UCR. ■

The following corollary is immediate from Theorem 1.

Corollary 1 *In the participation game under φ^E, every sequence of public goods levels at Nash equilibria converges to zero as an economy is replicated.*

Fixed contribution rules with budget balance. A *fixed contribution rule with budget balance* $\varphi^F = (\varphi_G^F, \varphi_T^F)$ is such that (i) $\varphi_T^F[\widetilde{\mu};\nu,C](u) = \varphi_T^F[\bar{\mu};\nu,C](u)$ for all (ν,C), all $\widetilde{\mu},\ \bar{\mu} \leq \nu$, and all $u \in Supp(\widetilde{\mu}) \cap Supp(\bar{\mu}) \subseteq Supp(\nu)$ and (ii) $\varphi_G^F[\mu;\nu,C] = C^{-1}\left(\sum_{u \in Supp(\widetilde{\nu})} \mu(u)\varphi_T^F[\mu;\nu,C](u)\right)$ for all (ν,C) and all $\mu \leq \nu$, where $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$.

Condition (i) of the fixed contribution rule with budget balance means that the fee of every agent is fixed and (ii) means that a public good is provided in a budget-balancing way. Under this rule, each agent always pays the same amount of a private good for a contribution. The public good is provided by using the total amount of contributions from the agents who contributed. The agents who do not pay the fee can free-ride the public good. Healy (2010) studies a voluntary participation game under this rule and analyzes when all agents voluntarily participate the mechanism (equilibrium participation).

We show that if a fixed contribution rule with budget balance does not satisfy UCR, then the public goods provision level monotonically increases as r goes up. Thus, under the rules, all sequences of equilibrium levels of the public good converge to zero as the economy is replicated.

Proposition 2 *Suppose that for all $\nu \in \mathcal{N}$, and all $u_i \in Supp(\nu)$, there exists $\bar{G}(\nu)$ such that $\varphi_G^F[\nu^r; \nu^r, C] < \bar{G}(\nu)$ for all r. Then, φ_G^F satisfies UCR.*

Proof. Suppose that φ_G^F satisfies UCR. Then, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $r \in \mathbb{Z}_{++}$ and all $\widetilde{\nu}^r, \bar{\nu}^r \in \mathcal{N}$ such that $\bar{\nu}^r \leq \widetilde{\nu}^r \leq \nu^r$ and $\frac{|\widetilde{\nu}^r - \bar{\nu}^r|}{|\nu^r|} \leq \delta$,

$$\left|\varphi_G^F[\widetilde{\nu}^r; \nu^r, C] - \varphi_G^F[\bar{\nu}^r; \nu^r, C]\right| < \epsilon$$

holds. Since φ_G^F is a fixed contribution rule with budget balance,

$$\sum_{u_i^r \in Supp(\nu^r)} \nu^r(u_i^r) \varphi_{\mathcal{T}}^F[\nu^r; \nu^r, C](u_i^r) = C\left(\varphi_G^F[\nu^r; \nu^r, C]\right) \leq C(\bar{G}(\nu)).$$

Thus, for all $u_i^r \in Supp(\nu^r)$, we have

$$\varphi_{\mathcal{T}}^F[\nu^r; \nu^r, C](u_i^r) \leq \frac{1}{\nu^r(u_i^r)} C(\bar{G}(\nu)) \leq \frac{1}{r\underline{\nu}} C(\bar{G}(\nu)),$$

where $\underline{\nu} = \min_{u_i \in Supp(\nu)} \nu(u_i)$. For all $\widetilde{\nu}^r, \bar{\nu}^r \in \mathcal{N}$ such that $\bar{\nu}^r \leq \widetilde{\nu}^r \leq \nu^r$ and

$\frac{|\widetilde{\nu}^r - \bar{\nu}^r|}{|\nu^r|} \leq \delta$, we have

$$
\begin{aligned}
& \sum_{u^r \in Supp(\nu^r)} \widetilde{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\widetilde{\nu}^r; \nu^r, C](u^r) - \sum_{u^r \in Supp(\nu^r)} \bar{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\bar{\nu}^r; \nu^r, C](u^r) \\
\leq \;& |\widetilde{\nu}^r - \bar{\nu}^r| \times \frac{1}{r\underline{\nu}} C(\bar{G}(\nu)) \\
= \;& \frac{|\widetilde{\nu}^r - \bar{\nu}^r|}{|\nu^r|} \times \frac{|\nu|}{\underline{\nu}} \times C(\bar{G}(\nu)) \\
\leq \;& \delta \times \frac{|\nu|}{\underline{\nu}} \times C(\bar{G}(\nu)).
\end{aligned}
$$

Since $C'(g)$ is increasing, and $\lim_{g \to 0} C'(g) = \underline{c}$, we have

$$
\begin{aligned}
& \varphi_G^F[\widetilde{\nu}^r; \nu^r, C] - \varphi_G^F[\bar{\nu}^r; \nu^r, C] \\
= \;& C^{-1}\left(\sum_{u^r \in Supp(\nu^r)} \widetilde{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\widetilde{\nu}^r; \nu^r, C](u^r) \right) - C^{-1}\left(\sum_{u^r \in Supp(\nu^r)} \bar{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\bar{\nu}^r; \nu^r, C](u^r) \right) \\
\leq \;& \frac{1}{\underline{c}} \left(\sum_{u^r \in Supp(\nu^r)} \widetilde{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\widetilde{\nu}^r; \nu^r, C](u^r) - \sum_{u^r \in Supp(\nu^r)} \bar{\nu}^r(u^r) \varphi_{\mathcal{T}}^F[\bar{\nu}^r; \nu^r, C](u^r) \right) \\
\leq \;& \delta \times \frac{1}{\underline{c}} \times \frac{|\nu|}{\underline{\nu}} \times C(\bar{G}(\nu)).
\end{aligned}
$$

Since $\frac{1}{\underline{c}} \times \frac{|\nu|}{\underline{\nu}} \times C(\bar{G}(\nu))$ is finite, for all $\epsilon > 0$, we can find $\delta > 0$ with $\delta \times \frac{1}{\underline{c}} \times \frac{|\nu|}{\underline{\nu}} \times C(\bar{G}(\nu)) < \epsilon$. Hence, UCR is satisfied for φ^F. ■

Note that the condition in the proposition is reasonable, if we require that φ^F achieves an individually rational allocation for all economies (the public good can be provided at more than efficient levels, but at least all agents get nonnegative utility levels).

Corollary 2 *Suppose that for all $\nu \in \mathcal{N}$, and all $u_i \in Supp(\nu)$, there exists $\bar{G}(\nu)$ such that $\varphi_G^F[\nu^r; \nu^r, C] < \bar{G}(\nu)$ for all r. In a voluntary participation game under φ^F, every sequence of the public goods level at Nash equilibria diminishes to zero as the economy is replicated.*

Healy (2010) treats a domain of continuous and monotone preferences; thus, his domain is more general than our domain. He focuses on a class of allocation rules that satisfy *equilibrium participation* (EP), which guarantees that all agents in the economy voluntarily choose to contribute the fee. When agents' preferences are quasi-linear, an allocation rule $\varphi = (\varphi_G, \varphi_T)$ satisfies EP if $u_i(\varphi_G[\nu; \nu, C]) - \varphi_T[\nu; \nu, C](u_i) \geq u_i(\varphi_G[\nu_{-i}; \nu, C])$ for all (ν, C) and all $u_i \in Supp(\nu)$. He shows that if the fixed contribution rule with budget balance satisfies EP, then every sequence of Nash equilibrium levels of a public good goes to zero as the economy is replicated following the definition by Milleron (1972). However, EP is rather strong: for example, the fixed contribution rule with efficiency does not satisfy EP.[5] With efficiency, the fee of an agent is relatively high, and given that many agents participate, the other agents do not gain by joining in the provision. Although Corollary 2 is a result on the quasi-linear preference domain, EP is not assumed. The convergence result is proven under all fixed contribution rules with budget balance.

A voluntary contribution rule. A voluntary contribution rule assigns

[5] In the fixed contribution rule with efficiency, the fixed fee is determined such that if all agents in the economy participate in the public good provision, the efficient level of the public good is provided.

a Nash equilibrium outcome of a voluntary contribution game, which is introduced by Bergstrom et al. (1986), for all (sub)economies. Before defining this allocation rule formally, we introduce some notation. Let $g(u) \equiv \arg\max_{g \geq 0} u(g) - C(g)$ for all $u \in \mathcal{U}$. Let $\bar{g}(\nu) \equiv \max_{u \in Supp(\nu)} g(u)$ and $\bar{u}(\nu) \in \arg\max_{u \in Supp(\nu)} g(u)$ for all $\nu \in \mathcal{N}$. For all (ν, C) and all $\widetilde{\nu} \leq \nu$, the voluntary contribution rule $\varphi^{VC} = (\varphi_G^{VC}, \varphi_T^{VC})$ assigns the following allocation:

$$\varphi_G^{VC}[\widetilde{\nu}; \nu, C] = \bar{g}(\widetilde{\nu}),$$
$$\varphi_T^{VC}[\widetilde{\nu}; \nu, C](\bar{u}(\widetilde{\nu})) = \frac{C\left(\bar{g}(\widetilde{\nu})\right)}{\widetilde{\nu}(\bar{u}(\widetilde{\nu}))} \text{ and } \varphi_T^{VC}[\widetilde{\nu}; \nu, C](u) = 0 \text{ for all } u \neq \bar{u}(\widetilde{\nu}).$$

Under φ^{VC}, the level of the public good is determined by $\bar{g}(\widetilde{\nu})$, which is the greatest amount of the public good which agents in $(\widetilde{\nu}, C)$ can provide individually (recall we assume quasi-linear utility). The cost of the public good is shared by participants with $\bar{u}(\widetilde{\nu})$ evenly. We show that φ_G^{VC} satisfies UCR.

Proposition 3 *Rule φ_G^{VC} satisfies UCR.*

Proof. We show that for all $(\nu, C) \in \mathcal{N}$ and all $\epsilon > 0$, there exists $r_\epsilon \in \mathbb{Z}_{++}$ such that $|\varphi_G^{VC}[\widetilde{\nu}^r; \nu^r, C] - \varphi_G^{VC}[\bar{\nu}^r; \nu^r, C]| < \epsilon$ for all $r \geq r_\epsilon$ and all $\widetilde{\nu}^r$, $\bar{\nu}^r$ such that $\widetilde{\nu}^r \leq \bar{\nu}^r \leq \nu^r$ and $\frac{|\widetilde{\nu}^r - \bar{\nu}^r|}{|\nu^r|} \leq \frac{1}{r_\epsilon |\nu|}$. If so, we complete the proof by letting $\delta = \frac{1}{r_\epsilon |\nu|}$. Suppose not. Then, there is a sequence $\{(\widetilde{\nu}^{t(r)}, \bar{\nu}^{t(r)})\}_{r=1}^{\infty}$ such that $t(r) \geq r$, $\frac{|\widetilde{\nu}^{t(r)} - \bar{\nu}^{t(r)}|}{|\nu^{t(r)}|} \leq \frac{1}{r|\nu|}$ and $|\varphi_G^{VC}[\widetilde{\nu}^{t(r)}; \nu^{t(r)}, C]-$

$\varphi_G^{VC}[\bar{\nu}^{t(r)}; \nu^{t(r)}, C]| \geq \epsilon$ are satisfied for all $r \in \mathbb{Z}_{++}$. However, $g(u^r)$ converges to zero as r goes to infinity for all u with $\nu(u) > 0$, which implies that $|\varphi_G^{VC}[\widetilde{\nu}^{t(r)}; \nu^{t(r)}, C] - \varphi_G^{VC}[\bar{\nu}^{t(r)}; \nu^{t(r)}, C]| \geq \epsilon$ is impossible for sufficiently large r. This is a contradiction. ■

The following corollary is an immediate implication of Theorem 1.

Corollary 3 *In a participation game with φ^{VC}, every sequence of public good levels at Nash equilibria diminishes to zero as the economy is replicated.*

5 Concluding Remarks

This paper has addressed how serious the free-riding problem in a voluntary participation game is. We examine under which allocation rules the Nash equilibrium level of a public good converges to zero as agents in the economy are replicated to large numbers. We introduce a continuity concept for a public goods provision rule, called UCR. We do not impose any condition on the cost sharing schemes, except for budget feasibility. If a public goods provision rule satisfies UCR, then every sequence of the equilibrium levels of a public good converges to zero through Milleron's (1972) replication. Several allocation rules, which have been studied in mechanism design theory, satisfy UCR. The efficient public goods provision rules such as the Lindahl and core allocation rules and the Clarke mechanism satisfy UCR. The construction of mechanisms whose equilibrium achieves allocative efficiency is one of the aims of implementation theory. Our result implies that if the economy consists of many agents and each agent can choose whether or not to participate in an efficient public goods mechanism, then the equilibrium level of a public good is inefficient and very small: it is hopeless to achieve allocative efficiency. The fixed contribution mechanism with budget balance and the voluntary contribution mechanism also satisfy UCR. Although our domain is restricted, in the domain of quasi-linear economies, our approach can treat some existing results as corollaries of our main theorem.

Finally, we note that there is an allocation rule outside the class of rules with UCR such that some sequence of equilibrium levels of a public good does not converge to zero.

Unanimous rule. Let $\varphi^U = (\varphi^U_G, \varphi^U_T)$ be a *unanimous rule*: for all $e = (\nu, C) \in \mathcal{E}$ and all $\widetilde{\nu} \leq \nu$,

$$\varphi^U_G[\widetilde{\nu}; \nu, C] = \begin{cases} \varphi^E_G[\nu; \nu, C] & \text{if } \widetilde{\nu} = \nu \\ 0 & \text{otherwise} \end{cases},$$

and

$$u_i(\varphi^U_G[\nu'; \nu, C]) - \varphi^U_T[\nu'; \nu, C](u_i) \geq 0 \text{ for all } \ u_i \in Supp(\nu').$$

Under this rule, a positive amount of a public good is produced if and only if all agents participate in the public good provision. The last condition is the individual rationality condition. Obviously, all agents participate at a Nash equilibrium even if the economy is replicated. Thus, there is a sequence of the levels of a public good at equilibrium which does not diminish to zero: even the efficient public good provision is achieved in an equilibrium sequence. Moreover, this rule assigns the same public good provision level for all replications (replication invariance). However, since φ^U has a discontinuity, it does not satisfy UCR. This unanimity rule shows that even if a rule satisfies many properties, if UCR is violated then the public goods provision level may not converge to zero.

References

[1] Bergstrom, T., L. Blume, and H. Varian (1986) On the Private Provision of Public Goods, *Journal of Public Economics*, Vol.29, pp. 25-49.

[2] Clarke, E. (1971) Multi-part Pricing of Public Goods, *Public Choice*, Vol.11, pp. 17-33.

[3] Conley, J.P. (1994) Convergence Theorems on the Core of a Public Goods Economy: Sufficient Conditions, *Journal of Economic Theory*, Vol.62, pp. 161-185.

[4] Dixit, A., and M. Olson (2000) Does Voluntary Participation undermine the Coase Theorem, *Journal of Public Economics*, Vol.76, pp. 309-335.

[5] Furusawa, T., and H. Konishi (2009) Contributing or Free-Riding?: Voluntary Participation in a Public Good Economy, Working Paper, Boston College.

[6] Groves T., and J. Ledyard (1977) Optimal Allocation of Public Goods: A Solution to the "Free Rider" Problem, *Econometrica*, Vol.45, pp. 783-811.

[7] Healy, P.J. (2010) Equilibrium Participation in Public Goods Allocations, *Review of Economic Design*, Vol.14, pp. 27-50.

[8] Hurwicz, L. (1979) Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points, *Review of Economic Studies*, Vol.46, pp. 217-225.

[9] Jackson, M.O. and T.R. Palfrey (2001) Voluntary Implementation, *Journal of Economic Theory*, Vol.98, pp. 1-25.

[10] Milleron, J.C. (1972) Theory of Value with Public Goods: A Survey Article, *Journal of Economic Theory*, Vol.5, pp. 419-477.

[11] Muench, T.J. (1972) The Core and the Lindahl Equilibrium of an Economy with a Public Good: An Example, *Journal of Economic Theory*, Vol.4, pp. 241-255.

[12] Palfrey, R.T., and H. Rosenthal (1984) Participation and the Provision of Discrete Public Goods: A Strategic Analysis, *Journal of Public Economics*, Vol.24, pp. 171-193.

[13] Saijo, T., and T. Yamato (1999) A Voluntary Participation Game with a Non-Excludable Public Good, *Journal of Economic Theory* Vol.84, pp. 227-242.

[14] Saijo, T., and T. Yamato (2010) Fundamental Impossibility Theorems on Voluntary Participation in the Provision of Non-excludable Public Goods, *Review of Economic Design*, Vol.14, pp. 51-73.

[15] Samuelson, P.A. (1954) The Pure Theory of Public Expenditure, *Review of Economics and Statistics*, Vol.36, pp. 387-389.

[16] Shinohara, R. (2007) Coalition-proof Equilibria in a Voluntary Participation Game, Working Paper, Shinshu University.

[17] Shinohara, R. (2009) The Possibility of Efficient Provision of a Public Good in Voluntary Participation Games, *Social Choice and Welfare*, Vol.32, pp. 367-387.

[18] Walker, M. (1981) A Simple Incentive Compatible Scheme for Attaining Lindahl Allocations, *Econometrica*, Vol.49, pp. 65-71.