Staff Paper Series ’09-03 October 2009

Coalition-proof Nash Equilibrium of Aggregative Games

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October 7, 2009

Abstract

The purpose of this paper is to clarify the properties of a coalition-proof Nash equilibrium in an aggregative game with monotone externality and strategic substitution. In this aggregative game, every Nash equilibrium satisfies the fundamental property that no coalition can deviate from the Nash equilibrium in such a way that all members of the coalition are better off and the deviation is self-enforcing. The three different characteristics of coalition-proof Nash equilibria are derived from this fundamental property: In this aggregative game, (i) some coalition-proof Nash equilibrium survives the iterative elimination of weakly dominated strategies, (ii) the set of coalition-proof Nash equilibria does not depend on which coalitions are feasible, and (iii) a coalition-proof Nash equilibrium provides the same outcome as a weak coalition equilibrium, which is a non-cooperative equilibrium concept that is based on some sort of farsightedness of players.

JEL Classification Numbers: C72.

Keywords: Coalition-proof Nash equilibrium; Aggregative game; Monotone externality; Strategic substitution

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1 Introduction

The purpose of this paper is to clarify the properties of a coalition-proof Nash equilibrium in an aggregative game with monotone externality and strategic substitution. Using a fundamental property of a Nash equilibrium in this game, we show that a coalition-proof Nash equilibrium has three distinct characteristics that have not been examined by earlier studies in this paper.

The coalition-proof Nash equilibrium is one of refinements of Nash equilibria, which was introduced by Bernheim et al. (1987). This equilibrium is immune to *self-enforcing* coalitional deviations. The self-enforceability of a coalitional deviation is such that no proper subcoalitions of the coalition can object to this coalitional deviation using their *self-enforcing* deviations. Hence, the notion of self-enforceability is defined recursively with respect to the number of members in a coalition. Due to this recursive nature of coalition-proof Nash equilibria, it is not easy to characterize this equilibrium and its properties have not been studied sufficiently.

In this paper, we focus on an *aggregative game* with *monotone externality* and *strategic substitution*. The aggregative game is such that the sets of the strategies of all players are the subsets of the real line and the payoff of each player depends on his/her strategy and on the sum of the strategies of the other players. Monotone externality requires that a switch in a player's strategies changes the payoffs of all the other players in the same direction. Strategic substitution means that the incentive to every player to reduce his/her strategy is preserved if the sum of the other players' strategies increases. A coalition-proof Nash equilibrium in an aggregative game with these two conditions was studied by Yi (1999) and Shinohara (2005). Yi (1999) showed that the set of coalition-proof Nash equilibria coincides with the (weakly) Pareto-efficient frontier of the set of Nash equilibria. Shinohara (2005) examined the relationship between the coalition-proof Nash equilibria and dominance relations. These results were proven on the basis of a fundamental property of a Nash equilibrium. This fundamental property is that *no group of players can deviate from every Nash equilibrium in such a way that all the players are better off by using their self-enforcing deviations in an aggregative game with these two conditions.* The other aspects of the coalition-proof Nash equilibrium are also shown based on this fundamental property. In this paper, we study the relationship between coalition-proof Nash equilibria and iterative elimination of *weakly* dominated strategies, the relationship between the equilibrium and feasible coalitions, and the relationship between the equilibrium and players' farsighted behavior. These three relationships are clarified on the basis of the above fundamental property.

As a preparation for the examination of these relations, we confirm that no Nash equilibrium is Pareto-dominated by any other Nash equilibrium in an aggregative game with monotone externality and strategic substitution. This, together with Yi (1999)'s result, implies that the set of Nash equilibria, the set of coalition-proof Nash equilibria, and the weakly Pareto-efficient frontier of the set of Nash equilibria coincide in this aggregative game (Corollary 1).

We first show that there is a coalition-proof Nash equilibrium that survives the iterative elimination of *weakly* dominated strategies in an aggregative game with monotone externality and strategic substitution. Obviously, the surviving coalition-proof Nash equilibrium consists of undominated strategies. As Peleg (1997) pointed out, a coalition-proof Nash equilibrium may consist of weakly dominated strategies and may be eliminated by the iterative weak dominance. He also showed that almost all dominant-strategy equilibria are coalition-proof; thus, such equilibria consist of undominated strategies. However, since there is not necessarily a dominant-strategy equilibrium in an aggregative game with monotone externality and strategic substitution, our results present another class of games in which there is a coalition-proof Nash equilibrium that survives the iterative domination procedure.

We also obtain that (i) every Nash equilibrium that consists of serially undominated strategies in the sense of weak domination is a coalition-proof Nash equilibrium of the original game and (ii) the serially undominated Nash equilibria do not Pareto-dominate each other. We also find that these statements do not necessariy hold true if an aggregative game does not satisfy one of the conditions of monotone externality and strategic substitution. These results show that the relationship between the coalition-proofness and the iterative elimination of weakly dominated strategies is completely different from the one between the coalition-proofness and the iterative elimination of ***strictly*** dominated strategies. Moreno and Wooders (1996) explored the relationship between the coalition-proof Nash equilibrium and the iterative elimination of strictly dominated strategies. They investigated a game with finite strategy sets and showed that if there exists a profile of serially undominated strategies that Pareto-dominates the other serially undominated strategies, then it is a coalition-proof Nash equilibrium. In contrast to the iterative strict domination, when the iterative elimination of weakly dominated strategies is adopted, a Pareto-superior serially undominated Nash equilibrium does not constitute a coalition-proof Nash equilibrium, which is demonstrated by our examples. Hence, some conditions need to be imposed on a game so that a result similar to that of the earlier study holds. Such conditions for an aggregative game are provided by our main result.

Second, a notion of *coalition-proof Nash equilibria with a restriction on coalition formation* is introduced to examine how the equilibrium outcomes depend on which coalitions are feasible. A *restriction on coalition formation* is a set of coalitions that can form in a game. A *coalition-proof Nash equilibria with a restriction on coalition formation* is stable against all self-enforcing deviations of coalitions that are in the restriction on coalition formation. Although coalition-proof Nash equilibria are generally different under distinct restrictions on coalition formation, we show that the sets of coalition-proof Nash equilibria are the same under any restriction of coalition formation in an aggregative game with monotone externality and strategic substitution. A result similar to this was provided by Serizawa (2006) in the mechanism design literature. Our analysis shows that the phenomenon similar to the one in Serizawa (2006) is observed in many games that have been frequently studied in economics.

Third, we show the equivalence between the coalition-proof Nash equilibria and an equilibrium that is based on the farsighted behavior of players, which is called a *weak coalition equilibrium*. This was introduced by Ju and Sarin (2009). The self-enforcing deviation of a coalition in the coalition-proof Nash equilibrium is robust only to the self-enforcing objections of the internal players in the coalition. However, self-enforceability does not impose any robustness on the deviations that non-members of the coalition join in. The coalition-proof Nash equilibrium does not suppose that players are farsighted either, because members of a coalition do not consider a sequence of successive deviations that are induced by the deviation of the coalition, as Xue (2000) and Ju and Sarin (2009) criticized. The weak coalition equilibrium is one of the equilibrium concepts that take the farsightedness of players and the non-internal deviations into account. The weak coalition equilibrium is a refinement of a Nash equilibrium and the set of weak coalition equilibria contains the set of coalition-proof Nash equilibria. Hence, the set of coalition-proof Nash equilibria and that of weak coalition equilibria do not necessarily coincide. However, we prove that these two equilibrium sets coincide in an aggregative game with monotone externality and strategic substitution. Thus, equilibrium outcomes do not change even if the farsighted stability concept of Ju and Sarin (2009) is investigated in these aggregative games.

2 Model

We consider a strategic game $G = [N,\, (X_i)_{i\in N},\, (u_i)_{i\in N}]$, where N is a finite set of players, X_i is the set of pure strategies of player i that is a subset of real numbers, and $u_i\colon \prod_{j\in N} X_j \to \mathbb{R}$ is the payoff function of player i. A group of players $S \subseteq N$ with $S \neq \emptyset$ is called a coalition. Denote the set of strategy profiles that can be chosen by S as $X_S \equiv \prod_{i\in S} X_i$ and denote $x_S \equiv (x_i)_{i\in S} \in X_S$, which is a strategy profile for S. The complement of S is denoted by $-S$. For notational simplicity, denote $X \equiv \prod_{j\in N} X_j$ and $x \equiv (x_j)_{j\in N} \in X$.

We focus on a class of games, which are called *aggregative games*. In aggregative games, the payoff of each player depends on his/her strategy and on the sum of the strategies of the other players.

Definition 1 A game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$ is an *aggregative game* if $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$ for every $i \in N$, every $x_i \in X_i$, and every x_{-i} and $x'_{-i} \in X_{-i}$ with $\sum_{j\in N\setminus\{i\}} x_j = \sum_{j\in N\setminus\{i\}} x'_j$.[1]

In this paper, we focus on the case in which all players choose only pure strategies.

The (pure-strategy) Nash equilibria are defined as usual. The set of (pure-strategy) Nash equilibria in G is denoted by $NE(G)$. In order to define a coalition-proof Nash equilibrium, *restricted games* are introduced. For any coalition $S \subseteq N$ and any strategy profile of the complement of S, $\bar{x}_{-S}$, denote the game restricted by $\bar{x}_{-S}$ by $G|\bar{x}_{-S}$ in which S is the set of players, X_S is the set of pure strategy profiles, and $u_i(\cdot,\, \bar{x}_{-S})\colon X_S \to \mathbb{R}$ is the payoff function of player $i \in S$. Now, the definition of a coalition-proof Nash equilibrium is provided as follows:

Definition 2 A *coalition-proof Nash equilibrium* $x^* \in X$ is defined inductively with respect to the number of members in coalitions:

(i) For every $i \in N$, x_i^* is a coalition-proof Nash equilibrium of $G|x^*_{-i}$ if x_i^* maximizes $u_i(\cdot, x^*_{-i})$.

[1] The original definition of an aggregative game is such that the payoff of a player depends on his/her strategy and the sum of the strategies of *all players*; on the other hand, we define an aggregative game by the game in which a payoff to a player depends on his/her strategy and the sum of the strategies of *the other players*. A class of aggregative games in the original definition is included in a class of our aggregative games. In this paper, we call a game in Definition 1 an aggregative game.

(ii) Let S be a coalition with $\#S \geq 2$. Assume that the coalition-proof Nash equilibria have been defined for every proper subset of S. Then, x^*_S is a coalition-proof Nash equilibrium of $G|x^*_{-S}$ if (a) and (b) are satisfied:
   (a) x^*_S is a *self-enforcing strategy profile* of $G|x^*_{-S}$, which is defined as follows: for every $T \subsetneq S$, x^*_T is a coalition-proof Nash equilibrium of $G|x^*_{-T}$.
   (b) No other self-enforcing strategy profile y_S of $G|x^*_{-S}$ Pareto-dominates x^*_S: $u_i(y_S, x^*_{-S}) > u_i(x^*_S, x^*_{-S})$ for every $i \in S$.

The set of coalition-proof Nash equilibria in G is denoted by $CPNE(G)$. For every $S \subseteq N$ and every $x_{-S} \in X_{-S}$, the set of coalition-proof Nash equilibria in a restricted game $G|x_{-S}$ is also denoted by $CPNE(G|x_{-S})$. Similarly, the set of Nash equilibria in a restricted game $G|x_{-S}$ is denoted by $NE(G|x_{-S})$.

In a coalition-proof Nash equilibrium of G, no proper coalition of N can not deviate in such a way that the coalition uses coalition-proof Nash equilibria of its corresponding restricted game and all members of the coalition are better off. Clearly, every coalition-proof Nash equilibrium is a Nash equilibrium.

3 Properties of Coalition-proof Nash Equilibrium

Coalition-proof Nash equilibria of aggregative games with Conditions 1 and 2 are examined in this section.

Condition 1 (Monotone externality) A game G satisfies *monotone externality* if either *positive externality* or *negative externality* is satisfied:

Positive externality. For all $i \in N$, all $x_i \in X_i$, and all x_{-i} and $\widehat{x}_{-i} \in X_{-i}$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \geq u_i(x_i, \widehat{x}_{-i})$ holds.

Negative externality. For all $i \in N$, all $x_i \in X_i$, and all x_{-i} and $\widehat{x}_{-i} \in X_{-i}$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \leq u_i(x_i, \widehat{x}_{-i})$ holds.

Condition 2 (Strategic substitution) For all $i \in N$, for all x_i, x'_i with $x_i > x'_i$, and for all x_{-i}, x'_{-i} with $\sum_{j \neq i} x_j > \sum_{j \neq i} x'_j$, if $u_i(x'_i, x'_{-i}) - u_i(x_i, x'_{-i}) \geq 0$, then $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) > 0$.

The condition of monotone externality requires that the payoff to player i changes monotonically with respect to the strategies of players other than i. The Cournot competition game is one of the examples that satisfy negative externality; the voluntary provision game of a (pure) public good is an example that satisfies positive externality.

The strategic substitution is as follows: Consider the situation in which player i does not have an incentive to choose x_i instead of x'_i when the other players choose x'_{-i}. Then, player i also has such an incentive if the other players increase their strategies from x'_{-i}.[2] Condition 2 is also satisfied by many games such as the Cournot competition game and the voluntary provision game of a pure public good.[3]

The property presented in Lemma 1 holds in an aggregative game with monotone externality and strategic substitution. This property is fundamental to clarify the characteristics of a coalition-proof Nash equilibrium in this paper.

Lemma 1 Suppose that an aggregative game G satisfies Conditions 1 and 2. For all $x^* \in NE(G)$, all non-empty $S \subseteq N$, and all $\widetilde{x}_S \in X_S$, if $u_i(\widetilde{x}_S, x^*_{-S}) > u_i(x^*)$ for every $i \in S$, then $\widetilde{x}_S$ is not a Nash equilibrium of $G|x^*_{-S}$.

Proof. Let $x^* \in NE(G)$. Let $S \subseteq N$ be a coalition. Consider a coalitional deviation from x^* by S in which S deviates from x^*_S to $\widetilde{x}_S$ and $u_i(\widetilde{x}_S, x^*_{-S}) > u_i(x^*_S, x^*_{-S})$ for each $i \in S$. We provide the proof in the case in which the positive externality condition is satisfied. The statement can then be shown in the case of the negative externality similarly.

It follows that $\sum_{j \neq S\setminus\{i\}} \widetilde{x}_j > \sum_{j \neq S\setminus\{i\}} x^*_j$ for every $i \in S$. If $\sum_{j \neq S\setminus\{i\}} \widetilde{x}_j \leq \sum_{j \neq S\setminus\{i\}} x^*_j$ for some $i \in S$, the following conditions are satisfied: $u_i(x^*_i, x^*_{-i}) \geq u_i(\widetilde{x}_i, x^*_{-i}) \geq u_i(\widetilde{x}_i, \widetilde{x}_{S\setminus\{i\}}, x^*_{-S})$. The first inequality of this condition follows from the definition of Nash equilibria and the second one follows from the positive monotonicity condition. The condition $u_i(x^*_i, x^*_{-i}) \geq u_i(\widetilde{x}_S, x^*_{-S})$ contradicts that the deviation by S is improving.

Summing up $\sum_{j \neq S\setminus\{i\}} \widetilde{x}_j > \sum_{j \neq S\setminus\{i\}} x^*_j$ for every $i \in S$ yields $\sum_{j \in S} \widetilde{x}_j > \sum_{j \in S} x^*_j$. Hence, there exists $k \in S$ such that $\widetilde{x}_k > x^*_k$. Since $\widetilde{x}_k > x^*_k$, $\sum_{j \neq S\setminus\{k\}} \widetilde{x}_j > \sum_{j \neq S\setminus\{k\}} x^*_j$, and $u_k(x^*_k, x^*_{-k}) - u_k(\widetilde{x}_k, x^*_{-k}) \geq 0$ for player k, we

[2] Strictly speaking, Condition 2 is weaker than the standard strategic substitution condition. The strategic substitution condition in Yi (1999) is defined as follows: For all $i \in N$, all x_i, x'_i, and all x_{-i}, x'_{-i}, if $x_i > x'_i$ and $\sum_{j \neq i} x_j > \sum_{j \neq i} x'_j$, then $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) > u_i(x'_i, x'_{-i}) - u_i(x_i, x'_{-i})$. Note that Yi (1999)'s strategic substitution implies Condition 2, but the converse is not true. Yi (1999)'s condition requires that $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i})$ is decreasing in x_{-i} for $x_i > x'_i$, while Condition 2 does not. This difference does not matter when proving our results.

[3] The other examples that satisfy these two conditions are provided by Yi (1999).

have $u_k(x_k^*, \widetilde{x}_{S\setminus\{k\}}, x_{-S}^*) - u_k(\widetilde{x}_k, \widetilde{x}_{S\setminus\{k\}}, x_{-S}^*) > 0$. This condition means that $\widetilde{x}_S$ is not a Nash equilibrium in $G|x_{-S}^*$. ■

Lemma 1 says that no coalition can deviate from a Nash equilibrium in such a way that every member of the coalition is better off and the coalitional deviation is self-enforcing in this aggregative game. From this property, Yi (1999) obtained the following proposition.

Proposition 1 (Yi, 1999) In aggregative games with Conditions 1 and 2, a profile of strategies is a coalition-proof Nash equilibrium if and only if it is a Nash equilibrium that is not strictly Pareto-dominated by any other Nash equilibrium.[4]

Applying Lemma 1 to the case of $S = N$, we obtain that a profile of strategies that Pareto-dominates a Nash equilibrium of G is not a Nash equilibrium. If there exist $x^* \in NE(G)$ and $y^* \in NE(G)$ such that x^* Pareto-dominates y^*, then x^* is not a Nash equilibrium from Lemma 1. Thus, no Nash equilibrium Pareto-dominates any other Nash equilibrium in an aggregative game with Conditions 1 and 2. This, together with Yi (1999)'s result, implies the following corollary.

Corollary 1 The set of Nash equilibria, the set of coalition-proof Nash equilibria, and the Pareto-efficient frontier of the set of Nash equilibria coincide in an aggregative game with Conditions 1 and 2.

Yi (1999) showed only the equivalence between the Pareto-superior Nash equilibrium and the coalition-proof Nash equilibrium. In addition to this, we confirm from Lemma 1 that the Nash equilibrium itself is equivalent to the coalition-proof Nash equilibrium. This has not been mentioned by any earlier study.

In the following subsections, we prove that a coalition-proof Nash equilibrium satisfies the three distinct properties that have not been reported by any existing literature. The statements are made based on Lemma 1.

3.1 Weakly Dominated Strategy and Coalition-proof Nash Equilibrium

In this subsection, we assume that X_i is a finite set for every $i \in N$.[5] The definition of weakly dominated strategies for a player is provided as follows:

[4] As mentioned in footnote 2, the strategic substitution in Yi (1999) is stronger than Condition 2. However, the statement can also be proven under Condition 2.

[5] We briefly mention the case of infinite strategy sets, later.

Definition 3 (Weakly dominated strategies) Let $Y_j \subseteq X_j$ for every $j \in N$. A strategy for $i \in N$, $y_i \in Y_i$, is a *weakly dominated* strategy in $\prod_{j \in N} Y_j$ if there is $z_i \in Y_i$ such that $u_i(z_i, y_{-i}) \geq u_i(y_i, y_{-i})$ for every $y_{-i} \in \prod_{j \neq i} Y_j$ and $u_i(z_i, y_{-i}) > u_i(y_i, y_{-i})$ for some $y_{-i} \in \prod_{j \neq i} Y_j$. A strategy y_i is an *undominated* strategy for i in $\prod_{j \in N} Y_j$ if y_i is not weakly dominated by any other strategy in Y_i.

We use the weak dominance relation based on pure strategies.[6] Note that the weak dominance relation is *transitive* and *asymmetric*. That is, for every $i \in N$ and for every $x_i, y_i, z_i \in Y_i \subseteq X_i$, if x_i weakly dominates y_i on $\prod_{j \in N} Y_j$ and y_i weakly dominates z_i on $\prod_{j \in N} Y_j$, then x_i weakly dominates z_i on $\prod_{j \in N} Y_j$ (transitivity) and, for every distinct $x_i, y_i \in Y_i \subseteq X_i$, if x_i weakly dominates y_i on $\prod_{j \in N} Y_j$, then y_i *does not* weakly dominate x_i (asymmetry).

Definition 4 (Iterated elimination of weakly dominated strategies) Let $X^0 \equiv \prod_{j \in N} X_j$. For every $i \in N$ and every $m \in \mathbb{Z}_{++}$, let X_i^m denote the set of strategies for i such that every $x_i \in X_i^{m-1} \backslash X_i^m$ is a weakly dominated strategy in $X^{m-1} \equiv \prod_{j \in N} X_j^{m-1}$. Suppose that at least one weakly dominated strategy is eliminated if weakly dominated strategies exist at each round of elimination. Let X^∞ be the set of strategy profiles such that no further strategy can be eliminated for every player.

For every game $G = [N,\, (X_i)_{i \in N},\, (u_i)_{i \in N}]$, let $G^m = [N,\, (X_i^m)_{i \in N},\, (u_i^m)_{i \in N}]$ ($m \in \mathbb{Z}_+ \cup \{\infty\}$) denote a game in which the set of strategy profiles is X^m and $u_i^m(x) = u_i(x)$ for every $i \in N$ and every $x \in X^m$. Hence, G^0 is the original game G and G^∞ is the game in which further elimination of weakly dominated strategies cannot be done.

Remark 1 A coalition-proof Nash equilibrium may consist of weakly dominated strategies; thus, a coalition-proof Nash equilibrium may be eliminated through the process of the iterative elimination of weak dominated strategies. The following example was presented by Peleg (1997), in which (A_1, B_1) is the unique coalition-proof Nash equilibrium and A_1 and B_1 are dominated strategies if $c_i > b_i$ and $a_i > c_i$ for every $i \in \{1, 2\}$. This equilibrium cannot survive the iterative elimination of weakly dominated strategies.

Transitivity and asymmetry, together with finiteness of strategy spaces, imply that every Nash equilibrium of G^{m+1} is also a Nash equilibrium of G^m for every $m \in \mathbb{Z}_+$, which is proven in Lemma 2.

[6] Börgers (1993) provided an interesting justification for pure-strategy weak dominance.

Table. 1 Coalition-proof Nash equilibrium consists of weakly dominated strategies

1 \ 2	B_1	B_2
A_1	a_1, a_2	b_1, a_2
A_2	a_1, b_2	c_1, c_2

Lemma 2 Suppose that the set of strategies is finite for every player. Then, (2.1) for every $i \in N$ and every $x_i \in X_i^m \backslash X_i^{m+1}$, there exists $x_i' \in X_i^{m+1}$ such that x_i' weakly dominates x_i in X^m, and (2.2) $NE(G^{m+1}) \subseteq NE(G^m)$ for every $m \in \mathbb{Z}_+$.

Proof. We first show (2.1). Suppose not. For some $i \in N$ and some $x_i \in X_i^m \backslash X_i^{m+1}$, no $x_i' \in X_i^{m+1}$ weakly dominates x_i. Since $x_i \in X_i^m \backslash X_i^{m+1}$, there is $x_i'' \in X_i^m$ such that x_i'' weakly dominates x_i. If $x_i'' \in X_i^{m+1}$, this is a contradiction since x_i'' weakly dominates x_i. Hence, x_i'' belongs to $X_i^m \backslash X_i^{m+1}$. Strategy x_i'' is also dominated in X^m; hence, there exists $x_i''' \in X_i^m$ such that x_i''' weakly dominates x_i''. Similarly, x_i''' is in $X_i^m \backslash X_i^{m+1}$ from the transitivity of weak dominance relation. This sort of dominance process continues. Since $X_i^m \backslash X_i^{m+1}$ is finite and the weak dominance relation is asymmetric, some strategy $\bar{x}_i \in X_i^m \backslash X_i^{m+1}$ exists such that $\bar{x}_i$ is not weakly dominated by any strategy in $X_i^m \backslash X_i^{m+1}$. However, $\bar{x}_i$ is a weakly dominated strategy in X^m, which implies that $\bar{x}_i$ must be weakly dominated by some $\widehat{x}_i \in X_i^{m+1}$. The transitivity of weak dominance relation imply that x_i is weakly dominated by $\widehat{x}_i \in X_i^{m+1}$, which is a contradiction.

Statement (2.2) is immediate from the definition of Nash equilibrium and (2.1). ■

Proposition 2 Suppose that the set of strategies is finite for every player. Suppose that an aggregative game G satisfies Conditions 1 and 2. Then, $CPNE(G^{m+1}) \subseteq CPNE(G^m)$ for every $m \in \mathbb{Z}_+$.

Proof. Suppose, to the contrary, that $x^* \notin CPNE(G^m)$ for some $x^* \in CPNE(G^{m+1})$ and some $m \in \mathbb{Z}_+$. Since $x^* \notin CPNE(G^m)$, there exists a coalition $S \subseteq N$ with strategy profile $\widetilde{x}_S \in CPNE(G|x_{-S}^*)$ such that $u_i(\widetilde{x}_S, x_{-S}^*) > u_i(x_S^*, x_{-S}^*)$ for every $i \in S$. Clearly, $\widetilde{x}_S$ belongs to $X_S^m \backslash X_S^{m+1}$ because $x^* \in CPNE(G^{m+1})$ and $x^* \in NE(G^m)$ from Lemma 2. It is straightforward from Lemma 1 that $\widetilde{x}_S$ is not a Nash equilibrium of $G^m|x_{-S}^*$, which is a contradiction. ■

Conditions 1 and 2 play an important role in the proof of Proposition 2. If one of these conditions fails, then the statement of the proposition does not hold as the following examples indicate.

Example 1 Consider the game in Table 2, which corresponds to the case of $a_1 = a_2 = 2$, $b_1 = b_2 = 0$, and $c_1 = c_2 = 1$ in Table 1. Let A_k and B_k $(k = 1, 2)$ be such that $A_k, B_k \in \mathbb{R}$, $A_1 > A_2$, and $B_1 > B_2$. In this case, the positive externality condition is satisfied, but strategic substitution is not. Clearly, A_1 and B_1 are dominated strategies. In the game after these strategies are eliminated, (A_2, B_2) is the unique coalition-proof Nash equilibrium, but this equilibrium is not coalition-proof in the original game.

Table. 2 Example 1

1 \ 2	B_1	B_2
A_1	2, 2	0, 2
A_2	2, 0	1, 1

Example 2 Consider a game in Table 3, in which $A_1 < A_2 < A_3$ and $B_1 < B_2 < B_3$.

Table. 3 Example 2

1 \ 2	B_1	B_2	B_3
A_1	0, 40	40, 40	40, 40
A_2	10, 41	45, 40	40, 35
A_3	20, 38	50, 30	40, 20

This game satisfies strategic substitution because

$$\begin{aligned}
&u_1(A_1,B_3)-u_1(A_3,B_3)=0>u_1(A_1,B_2)-u_1(A_3,B_2)=-10\\
>&\ u_1(A_1,B_1)-u_1(A_3,B_1)=-20,\\
&u_1(A_1,B_3)-u_1(A_2,B_3)=0>u_1(A_1,B_2)-u_1(A_2,B_2)=-5\\
>&\ u_1(A_1,B_1)-u_1(A_3,B_1)=-10,\ \text{and}\\
&u_1(A_2,B_3)-u_1(A_3,B_3)=0>u_1(A_2,B_2)-u_1(A_3,B_2)=-5\\
>&\ u_1(A_2,B_1)-u_1(A_3,B_1)=-10.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&u_2(A_3,B_1)-u_2(A_3,B_2)=8>u_2(A_2,B_1)-u_2(A_2,B_2)=1\\
>&\ u_2(A_1,B_3)-u_2(A_1,B_3)=0,\\
&u_2(A_3,B_1)-u_2(A_3,B_3)=18>u_2(A_2,B_1)-u_2(A_2,B_3)=6\\
>&\ u_2(A_1,B_1)-u_2(A_2,B_3)=0,\ \text{and}\\
&u_2(A_3,B_2)-u_2(A_3,B_3)=10>u_2(A_2,B_2)-u_2(A_2,B_3)=5\\
>&\ u_2(A_1,B_2)-u_2(A_1,B_3)=0.
\end{aligned}$$

For player 1, it is satisfied that $u_1(A_2,B_1)<u_1(A_2,B_2)$ and $u_1(A_3,B_1)<u_1(A_2,B_2)$; a similar condition holds for player 2. Hence, this game does not satisfy the monotone externality. In this game, (A_1,B_3) is the only coalition-proof Nash equilibrium but both A_1 and B_3 are weakly dominated strategies for players 1 and 2, respectively. Strategies A_2 and B_2 are also weakly dominated strategies. After the elimination of A_2, A_3, B_2, and B_3, the only surviving strategy profile is (A_3,B_1). This is trivially a coalition-proof Nash equilibrium in a game after these strategies are eliminated, but this is not coalition-proof in the original game.

It follows from Proposition 2 that every coalition-proof Nash equilibrium of G^∞ is a coalition-proof Nash equilibrium of G. Thus, there is a coalition-proof Nash equilibrium of G that is not eliminated by iterative weak dominance. Of course, such a coalition-proof Nash equilibrium consists of undominated strategies. From Corollary 1 and Proposition 2, we have the following corollary.

Corollary 2 Suppose that an aggregative game G satisfies Conditions 1 and 2 and the strategy sets for all players are finite. Then, every Nash equilibrium of G^∞ is a coalition-proof Nash equilibrium of G.

Proof. First, note that Conditions 1 and 2 hold in G^m for every $m \in \mathbb{Z}_{++} \cup \{\infty\}$ if these conditions are satisfied in G. Applying Corollary 1 to G^∞ yields $NE(G^\infty) = CPNE(G^\infty)$. From Proposition 2, $CPNE(G^\infty) \subseteq CPNE(G)$. ■

Corollary 2 shows that the relationship between coalition-proofness and the iterative elimination of weakly dominated strategies is completely different from the one between coalition-proofness and the iterative elimination of *strictly* dominated strategies. Moreno and Wooders (1996) examined the relationship between the coalition-proof Nash equilibrium and the iterative elimination of strictly dominated strategies. They treated a game with finite strategy sets and showed that if there exists a profile of serially undominated strategies that Pareto-dominates all other serially undominated strategies, then this is a coalition-proof Nash equilibrium in the game. Milgrom and Roberts (1996) extended their result to a game with infinite strategy spaces under the condition of strategic complementarity. These two papers also showed that a profile of serially undominated strategies is the unique coalition-proof Nash equilibrium in a dominance solvable game, in which the serially undominated strategy profile is uniquely determined.

However, when we consider the elimination of weakly dominated strategies, serially undominated strategy profiles are not necessarily coalition-proof. The difference between iterative weak domination and iterative strict domination is very noticable in dominance solvable games. While the unique profile of serially undominated strategies is a coalition-proof Nash equilibrium when iterative strict dominance is analyzed, this is not necessarily observed when iterative weak dominance is considered. In Example 1, (A_2, B_2) is the unique strategy profile that consists of serially undominated strategies in the sense of weak domination, but this is not coalition-proof in the game. This applies to Example 2. In general, Pareto-superior serially undominated Nash equilibrium is not coalition-proof if an aggregative game fails to satisfy one of the conditions of monotone externality and strategic substitution. Corollary 2 presented a sufficient condition for an aggregative game under which some serially undominated strategy profile in the sense of weak domination is coalition-proof in the game, analogous to the iterative elimination of strictly dominated strategies.

Finally, we remark several points. The first is related to the assumption of finite strategy spaces. The condition for finite strategy spaces is used only in the proof of Lemma 2. The assumption of finite strategy sets guarantees that every weakly dominated strategy is weakly dominated by an **undominated** strategy. From this property, it follows that every Nash equilibrium of G^{m+1} is also a Nash equilibrium of G^m for every m. However, when the strategy sets are infinite, a strategy may be weakly dominated by another strategy that is weakly dominated by the other weakly dominated strategy and so on. Therefore, in the case of infinite strategy sets, some Nash equilibrium of G^{m+1} may not be a Nash equilibrium of G^m for some m due to such an infinite sequence of dominance relations, which means that our main results may not hold in the case of infinite strategy spaces.[7] However, results that are similar to our main results can be obtained even in the case of infinite strategy spaces if we adopt a variant of iterative weak dominance in which a strategy is eliminated only if it is weakly dominated by an **undominated strategy**. Under this iterative dominance concept, every eliminated strategy has an undominated strategy that weakly dominates the eliminated strategy; hence, it follows that every Nash equilibrium of G^{m+1} is also a Nash equilibrium of G^m for every m in games with infinite strategy sets.

Second, from Proposition 2, it follows that a coalition-proof Nash equilibria of G^∞ are coalition-proof Nash equilibria of the original game G. However, not all of the coalition-proof Nash equilibrium in G survive the iterative elimination of weakly dominated strategies. The following example shows that a coalition-proof Nash equilibrium of G^m may not be coalition-proof in G^{m+1} for some $m \in \mathbb{Z}_+$. Some coalition-proof Nash equilibrium in G^m may be eliminated through iterative weak dominance.

Example 3 Consider a game in Table 4. Let s_i^* and $\widetilde{s}_i$ be in $\mathbb{R}$ such that $s_i^* > \widetilde{s}_i$ for every $i \in N$. This game satisfies the negative externality condition since $u_1(s_1, s_2^*) < u_1(s_1, \widetilde{s}_2)$ for every $s_1 = s_1^*, \widetilde{s}_1$ and $u_2(s_1^*, s_2) < u_2(\widetilde{s}_1, s_2)$ for every $s_2 = s_2^*, \widetilde{s}_2$. The strategic substitutability also holds because $u_1(\widetilde{s}_1, s_2^*) - u_1(s_1^*, s_2^*) > u_1(\widetilde{s}_1, \widetilde{s}_2) - u_1(s_1^*, \widetilde{s}_2) = 0$ and $u_2(\widetilde{s}_1, \widetilde{s}_2) < u_2(\widetilde{s}_1, s_2^*)$. In this game, $(\widetilde{s}_1, s_2^*)$ and $(s_1^*, \widetilde{s}_2)$ are coalition-proof Nash equilibria. However, s_1^* and $\widetilde{s}_2$ are both weakly dominated strategies. Hence, the coalition-proof Nash equilibrium $(s_1^*, \widetilde{s}_2)$ is eliminated, while $(\widetilde{s}_1, s_2^*)$ survives the elimination of weakly dominated strategies.

Table. 4 Example 3

1 \ 2	s_2^*	$\widetilde{s}_2$
s_1^*	$-3,\ 0$	$0,\ 0$
$\widetilde{s}_1$	$-2,\ 2$	$0,\ 1$

[7] There are economic games in which the infinite sequence of dominance relations does not occur. The standard Cournot game and the public-good-provision game are such examples.

Third, one may think that it is meaningless to examine the relationship between coalition-proofness and the elimination of weakly dominated strategies because the iterative elimination of weakly dominated strategies is an order-dependent procedure. However, our results can be applied to an order-independent procedure that was proposed by Marx and Swinkels (1997) and Marx (1999), called the *elimination of nicely weakly dominated strategies.* This scheme is defined as follows: For every player i, a strategy x_i *nicely weak dominates* a strategy x'_i on X if (i) x_i weakly dominates x'_i on X and (ii) for all $x_{-i} \in X_{-i}$ if $u_i(x_i, x_{-i}) = u_i(x'_i, x_{-i})$, then $u_j(x_i, x_{-i}) = u_j(x'_i, x_{-i})$ for every $j \in N$. The concept of nice weak dominance is also defined for $Y \subseteq X$. Eliminating the nicely weakly dominated strategies, we can construct a sequence of games $\{G^m\}_{m=0}^{\infty}$ as in the standard iterative weak domination. For all $i \in N$ and all i's strategies x_i and $\widetilde{x}_i$, x_i is weakly dominated by $\widetilde{x}_i$ if x_i is nicely weakly dominated by $\widetilde{x}_i$. This property, together with the finiteness of strategy sets, implies that $NE(G^{m+1}) \subseteq NE(G^m)$ for all $m \in \mathbb{Z}_+$. By using Lemma 1, we can show that $CPNE(G^{m+1}) \subseteq CPNE(G^m)$ under Conditions 1 and 2 for all m.

3.2 Coalition-proof Nash Equilibrium under Restricted Coalition Formation.

What happens if the process of coalition formation is restricted for some reason? For example, the coordination of many agents is very costly and it is difficult for a large coalition to be formed, some individuals are unable to form a coalition because of geographical compulsions, firms are prohibited to form a cartel by law, etc. We examine this questions in an aggregative game with Conditions 1 and 2 by introducing a *coalition-proof Nash equilibrium with a restriction on coalition formation.*

A *restriction on coalition formation* is defined as a subset of $2^N \backslash \{\emptyset\}$. Let $\mathcal{T}$ denote a restriction of coalition formation. Suppose that only the coalitions in $\mathcal{T}$ can be formed. We call an element in $\mathcal{T}$ a *feasible coalition.* Since singleton coalitions can always switch strategy without cooperation with the other players, we assume that every singleton coalition belongs to $\mathcal{T}$. One of the examples of the restrictions on coalition formation is $\mathcal{T}^k \equiv \{S \subseteq N | \#S \leq k\}$ for some $k \leq n$: In this restriction, at most k players can coordinate their strategies.[8]

[8] The first paper to introduce this type of restriction on coalition formation to non-cooperative equilibrium concepts is Deb et al. (1997). They applied this restriction to Aumann (1959)'s strong Nash equilibrium and introduced a strong Nash equilibrium with $\mathcal{T}^k$. Serizawa (2006) also used a similar restriction in the mechanism design literature.

Definition 5 A strategy profile x^* is a coalition-proof Nash equilibrium with a restriction on coalition formation $\mathcal{T}$ in G is defined with recursion in the number of players in coalitions.

(i) For every i, x_i^* is a coalition-proof Nash equilibrium with $\mathcal{T}$ of $G|x_{-i}^*$ if $x_i^* \in \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}^*)$.

(ii) Let $S \in \mathcal{T}$ be a feasible coalition with $\#S \geq 2$. Assume that a coalition-proof Nash equilibrium with $\mathcal{T}$ of $G|x_{-T}^*$ has been defined for every $T \subsetneq S$ such that $T \in \mathcal{T}$. Then, x_S^* is a coalition-proof Nash equilibrium with $\mathcal{T}$ of $G|x_{-S}^*$ if the following two conditions are satisfied: (i) for every $T \subsetneq S$ with $T \in \mathcal{T}$, x_T^* is a coalition-proof Nash equilibrium with $\mathcal{T}$ of $G|x_{-T}^*$ (self-enforceability with $\mathcal{T}$ in $G|x_{-S}^*$) and (ii) there is no $y_S \in X_S$ such that y_S is self-enforcing with $\mathcal{T}$ in $G|x_{-S}^*$ and $u_i(y_S, x_{-S}^*) > u_i(x_S^*, x_{-S}^*)$ for every $i \in S$.

The coalition-proof Nash equilibrium with $\mathcal{T}$ is equivalent to the Nash equilibrium if $\mathcal{T} = \{\{i\} | \, i \in N\}$ and this is equivalent to the coalition-proof Nash equilibrium in Definition 2 if $\mathcal{T} = 2^N \backslash \{\emptyset\}$. For every $\mathcal{T}$, every coalition-proof Nash equilibrium with $\mathcal{T}$ is a Nash equilibrium. For all $\mathcal{T}, \widehat{\mathcal{T}} \subseteq 2^N \backslash \{\emptyset\}$, if $\mathcal{T} \subseteq \widehat{\mathcal{T}}$, then every coalition-proof Nash equilibrium with $\widehat{\mathcal{T}}$ is a coalition-proof Nash equilibrium with $\mathcal{T}$. If $\mathcal{T} \neq \widehat{\mathcal{T}}$, coalition-proof Nash equilibria with $\mathcal{T}$ and $\widehat{\mathcal{T}}$ generally assign different strategies. However, in an aggregative game with Conditions 1 and 2, the sets of coalition-proof Nash equilibria with different restrictions of coalition formation coincide.

Proposition 3 For all $\mathcal{T}, \widehat{\mathcal{T}} \subseteq 2^N \backslash \{\emptyset\}$, a strategy profile is a coalition-proof Nash equilibrium with $\mathcal{T}$ if and only if it is a coalition-proof Nash equilibrium with $\widehat{\mathcal{T}}$ in an aggregative game with Conditions 1 and 2.

Proof. Let $x^* \in X$ be a coalition-proof Nash equilibrium with $\mathcal{T}$. Suppose that x^* is not a coalition-proof Nash equilibrium with $\widehat{\mathcal{T}}$. Then, there exists a coalition $S \in \widehat{\mathcal{T}} \backslash \mathcal{T}$ with strategy profile $\widetilde{x}_S \in X_S$ such that $\widetilde{x}_S$ is a coalition-proof Nash equilibrium with $\mathcal{T}$ in $G|x_{-S}^*$ and $u_i(\widetilde{x}_S, x_{-S}^*) > u_i(x^*)$ for every $i \in S$. Notice that $\widetilde{x}_S \in NE(G|x_{-S}^*)$. However, it is straightforward from Lemma 1 that $\widetilde{x}_S \notin NE(G|x_{-S}^*)$. This is a contradiction. The converse can be proven similarly. ■

An implication of Proposition 3 is that a restriction on coalition formation does not affect the equilibrium outcomes in an aggregative game with Conditions 1 and 2. For example, consider the Cournot oligopoly game, which is an aggregative game that satisfies Conditions 1 and 2. Consider a situation in which firms form a cartel and coordinate their quantities but only small cartels can be formed because the coordination of many firms is difficult and costly. Our results imply that the equilibrium outcomes against self-enforcing cartel behavior in this situation is the same as those in the situation in which all possible coalitions are feasible. Thus, in the aggregative game, the self-enforcing behavior of cartels leads to the same outcomes, irrespective of how many players can coordinate their strategies jointly.

Serizawa (2006) studied a coalition-proof Nash equilibrium under a restriction on coalition formation in which at most two players can form a coalition in the mechanism design literature. He introduced the axiom of *effective pairwise strategy-proofness* and examined which social choice rules satisfy this axiom. The effective strategy-proofness requires that a truth-telling dominant-strategy equilibrium should be a coalition-proof Nash equilibrium with $\mathcal{T}^2 = \{S \subseteq N | 1 \leq \#S \leq 2\}$ in the preference revelation game. Thus, a social choice rule with this axiom is immune to unilateral manipulation and self-enforcing pairwise manipulation of preferences. The effective pairwise strategy-proofness seems to be much weaker than the axiom of *group strategy-proofness*, which requires that no groups of players should manipulate their preferences. However, Serizawa (2006) showed the equivalence between the effective pairwise strategy-proofness and the group strategy-proofness in some environments such as an economy with public goods. This implies that the truth-telling strategy profile is a coalition-proof Nash equilibrium with $\mathcal{T}^2$ if and only if it is a coalition-proof Nash equilibrium with $\mathcal{T}^n = \{S \subseteq N | 1 \leq \#S \leq n\}$ in the revelation game. Of course, the revelation game is not an aggregative game and also does not satisfy Conditions 1 and 2. Our analysis shows that the phenomenon similar to the one in Serizawa (2006) is observed in many non-cooperative games that have been frequently studied in economics.

3.3 Coalition-proofness and Farsighted Stability

Ju and Sarin (2009) addressed the following question: What is a "reasonable" self-enforcing coalition deviation? They introduced a new concept of non-cooperative equilibrium that is based on the coalition-proof Nash equilibrium. The coalition-proof Nash equilibrium is immune to self-enforcing deviations defined in Definition 2. The self-enforcing deviations are such that proper subcoalitions of a coalition S do not deviate in a self-enforcing way after a deviation of S. Thus, the coalition-proof Nash equilibrium assumes that only the internal members of S object to the deviation of S and outside players of S do not respond to this deviation. This assumption is valid when a deviation of a coalition S is known only to the members of S. However, this may not be natural in a situation in which a deviation of S is open to players outside

S and outsiders can respond to this deviation. A *weak coalition equilibrium*, which was introduced by Ju and Sarin (2009), is one of equilibrium concepts that consider such a situation.

The weak coalition equilibrium is immune to all *strictly self-enforcing deviations*. The strictly self-enforcing deviations are defined as coalitional deviations which satisfy not only (*internal*) self-enforceability in the sense of Definition 2 but also *across-coalitional self-enforceability*. The notion of across-coalitional self-enforceability is based on the farsightedness of players and the following situation is considered: A deviation of coalition S from a strategy profile may induce a deviation of the other coalition, which also leads to further coalitional deviations and so on. A coalition deviates if this deviation benefits all members of the coalition at the consequence of successive deviations. This notion of deviations admits non-internal deviations. For example, after a deviation by coalition S, a subcoalition of S and players outside S may deviate jointly or players outside S form a coalition and make a deviation.

The formal definition of weak coalition equilibria are presented as follows: A *sequence of deviations* from $x \in X$ to $y \in X$, which is denoted by $\{(x^r, S^r)\}_{r=0}^{m \in \mathbb{Z}_{++}}$, is defined as follows: $x^0 \equiv x$, $S^0 \equiv \emptyset$, $x^m \equiv y$, and $x^r \in X$, $S^r \subseteq N$, and $x^r \equiv (x^r_{S^r}, x^{r-1}_{-S^r})$ for every $r \geq 1$; this represents that S^r deviates from x^{r-1} by using $x^r_{S^r}$. *A sequence of internally self-enforcing deviations* from x to y is a sequence of deviations such that $u_i(y) > u_i(x^r)$ for any $i \in S^r$ and $x^r_{S^r} \in CPNE(G|x^{r-1}_{-S^r})$ for every $r \geq 1$.

The formal definitions of strictly self-enforcing deviations and weak coalition equilibria are as follows:

Definition 6 A strategy profile $\widetilde{x}_S \in X_S$ is a *strictly self-enforcing deviation* of S from x if it satisfies

(C1) $u_i(\widetilde{x}_S, x_{-S}) > u_i(x)$ for every $i \in S$,

(C2) $\widetilde{x}_S \in CPNE(G|x_{-S})$, and

(C3) $\widetilde{x}_S$ is an *across-coalitionally self-enforcing deviation* of S from x: there is a sequence of internally self-enforcing deviations $\{(x^r, S^r)\}_{r=0}^{m}$ such that $S^1 = S$, $x^1 = (\widetilde{x}_S, x_{-S})$, $x^m \in CPNE(G)$, and $u_i(x^m) > u_i(x)$ for any $i \in S$.

Definition 7 A *weak coalition equilibrium* is a Nash equilibrium that is immune to all strictly self-enforcing deviations.

Across-coalitional self-enforceability indicates that players coordinate their strategies in the absence of binding agreements; hence, an agreement $x^r_{S^r}$ of a coalition S^r ($r \geq 1$) must be internally self-enforcing if players S^r reach an agreement, and the final outcome x^m must be a coalition-proof Nash equilibrium since no further deviation can occur from the final outcome. The difference between the coalition-proof Nash equilibrium and the weak coalition equilibrium lies in condition (C3). This condition restricts the self-enforcing deviations in the sense of Definition 2. The weak coalition equilibrium assumes that a coalition deviates if and only if the coalition has an internally self-enforcing deviation that results in improvement at the terminal point of successive deviations.

Example 4 Consider the following example depicted in Table 5, in which player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. The first entry in each box is player 1's payoff, the second is player 2's, and the third is player 3's. In this game, (A_1, B_1, C_1) is a weak coalition equilibrium that is not coalition-proof and (A_3, B_2, C_2) is a coalition-proof Nash equilibrium. The equilibrium (A_1, B_1, C_1) is not coalition-proof since a deviation by players 1 and 2 from (A_1, B_1) to (A_3, B_2) is self-enforcing in the sense of Definition 2. However, this deviation is not strictly self-enforcing because player 3 wants to switch from C_1 to C_3 after the deviation by players 1 and 2, which finally leads to a coalition-proof Nash equilibrium (A_3, B_2, C_2). At the final outcome (A_3, B_2, C_2), players 1 and 2 are worse off.

Table. 5 Payoff matrix of Example 4

1 \ 2	B_1	B_2
A_1	2, 2, 2	0, 0, 1
A_2	2, 0, 1	2, 0, 1
A_3	0, 0, 1	3, 3, 0

3 C_1

1 \ 2	B_1	B_2
A_1	0, 0, 0	1, 1, 0
A_2	0, 0, 2	0, 2, 0
A_3	0, 0, 0	1, 1, 3

3 C_2

Trivially, any weak coalition equilibrium is a Nash equilibrium. Any coalition-proof Nash equilibrium is a weak coalition equilibrium because the coalition-proof Nash equilibrium is stable against all self-enforcing deviations, deviations with (C1) and (C2), while the weak coalition equilibrium is immune to the deviations with all the three conditions. Thus, the weak coalition equilibrium is more likely to exist than the coalition-proof Nash equilibrium. In fact, the weak coalition equilibrium exists if a Nash equilibrium exists, as Ju and Sarin (2009) proved, while a coalition-proof Nash equilibrium does not necessarily exist even if a Nash equilibrium exists. Hence, there is a weak coalition equilibrium in many games studied in economics.

Although they showed the existence of weak coalition equilibria, it is open as to which Nash equilibrium is a weak coalition equilibrium. Here, we provide a characterization of the equilibria in an aggregative game under Conditions 1 and 2 by using Lemma 1.

Proposition 4 A strategy profile is a weak coalition equilibrium if and only if it is a coalition-proof Nash equilibrium in an aggregative game with Conditions 1 and 2.

Proof. Obviously, every coalition-proof Nash equilibrium is a weak coalition equilibrium. We show the converse. Let $x \in X$ denote a weak coalition equilibrium. Hence, x is a Nash equilibrium in G. From Lemma 1 it follows that there is no coalitional deviation that satisfies (C1) and (C2). This means that x is a coalition-proof Nash equilibrium. ■

In aggregative games with Conditions 1 and 2, no coalitions can deviate from a Nash equilibrium in a way that satisfies coalitional profitability (C1) and (internal) self-enforceability (C2). Therefore, there is no coalition deviation that satisfies from (C1) to (C3) trivially. This is the intuition behind Proposition 4.

The following is a remark for Proposition 4: In the across-coalitional self-enforceability, a sequence $\{(x^r, S^r)\}_{r=0}^{m}$ is assumed to satisfy $x^r \in CPNE(G|x^{r-1}_{-S^r})$ for every r and $x^m \in CPNE(G)$. Proposition 4 still holds if we replace this assumption with the assumption that $x^r \in NE(G|x^{r-1}_{-S^r})$ for every r and $x^m \in NE(G)$, which is another conceivable assumption for the agreements of coalitions and the final outcomes without binding agreements. There is the possibility that the set of weak coalition equilibria shrinks when a coalition-proof Nash equilibrium is replaced with a Nash equilibrium. However, the equilibrium set does not change by this replacement since (C1) is incompatible with (C2) in an aggregative game with Conditions 1 and 2.

Ju and Sarin (2009) presented examples in which a weak coalition equilibrium exists but a coalition-proof Nash equilibrium does not. Hence, the weak coalition equilibirum assigns a strategy profile different from the coalition-proof Nash equilibrium. However, in the aggregative games with Conditions 1 and 2, these two equilibria are equivalent. The class of aggregative games with Conditions 1 and 2 includes games that have been studied in economics such as the standard Cournot game and the public good provision game. We find that there is no difference between these two equilibria in some games that have been frequently studied in economics.

4 Concluding Remarks

We have investigated a coalition-proof Nash equilibrium of an aggregative game with monotone externality and strategic substitution. The concept of coalition-proof Nash equilibrium is defined with recursion in the number of players in a coalition. Due to the recursive nature of this equilibrium, it is difficult to examine which properties are satisfied by the coalition-proof Nash equilibrium even in a restricted class of games such as aggregative games. First, we showed that no improving coalitional deviations from every Nash equilibrium are self-enforcing in aggregative games with monotone externality and strategic substitution. Using this property, we proved that (i) some coalition-proof Nash equilibrium survives the iterative elimination of weak dominated strategies, (ii) the sets of coalition-proof Nash Equilibria are the same under any restriction on coalition formation, and (iii) the coalition-proof Nash equilibrium is equivalent to the weak coalition equilibrium. Statement (i) implies that there exist serially undominated strategies in the sense of weak domination that is a coalition-proof Nash equilibrium. Statement (ii) means that coalition-proof outcomes in this type of games do not depend on which coalitions are feasible. Statement (iii) indicates that the outcomes in equilibria do not change even if players behave according to some farsighted behavioral principle. Since many examples of aggregative games with the two conditions exist in economics, our result might be useful to specify the properties of a coalition-proof Nash equilibrium in economic analyses.

Finally, we make comments on the topics for future research. In general, how to restrict feasible coalitions affects the coalition-proof Nash equilibria with restrictions on coalition formation. However, in games other than our games, it is open as to how the equilibrium set changes as the restriction on coalition formation changes. This is left for the future. Several authors such as Mariotti (1997) and Xue (2000) have introduced concepts of farsighted stability different from the weak coalition equilibrium. However, it is unknown as to which strategy profiles satisfy these farsighted stability concepts in an aggregative game with monotone externality and strategic substitution. It would be interesting to investigate this.

Acknowledgements. I gratefully acknowledge financial support from Grant-in-Aid for Young Scientists (B) from the Japan Society for Promotion of Science. Needless to say, any remaining errors are my own.

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