解答例

付録 A

問 A.1　期待値の定義より,

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xp(x)dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx \\ &= -\left[xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{1}{\lambda}, \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 p(x)dx = \int_0^\infty \lambda x^2 e^{-\lambda x}dx \\ &= -\left[x^2 e^{-\lambda x}\right]_0^\infty + \int_0^\infty 2xe^{-\lambda x}dx \\ &= \left[-2x\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}dx \\ &= \frac{2}{\lambda}\left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{2}{\lambda^2} \end{aligned}$$

なので，第 1 章の定理 7 の (1) より，$V[X] = E[X^2] - E[X]^2 = \dfrac{1}{\lambda^2}$.

問 A.2　(1)　X_1 と X_2 は独立なので,

$$\begin{aligned} P(X_1 + X_2 = i) &= \sum_{j=0}^{i} P(X_1 = j)P(X_2 = i - j) \\ &= \sum_{j=0}^{i} {}_{n_1}\mathrm{C}_j\, p^j (1-p)^{n_1 - j} \times {}_{n_2}\mathrm{C}_{i-j}\, p^{i-j}(1-p)^{n_2 - i + j} \\ &= \sum_{j=0}^{i} {}_{n_1}\mathrm{C}_j \cdot {}_{n_2}\mathrm{C}_{i-j}\, p^i (1-p)^{n_1+n_2-i} \end{aligned}$$

となる．ここで,

$$\sum_{j=0}^{i} {}_{n_1}\mathrm{C}_j \cdot {}_{n_2}\mathrm{C}_{i-j} = {}_{n_1+n_2}\mathrm{C}_i$$

なので,

$$P(X_1 + X_2 = i) = {}_{n_1+n_2}\mathrm{C}_i\, p^i (1-p)^{n_1+n_2-i}$$

を得る．よって，$X_1 + X_2$ は二項分布 $B(n_1 + n_2, p)$ に従う.

(2)　X_1 と X_2 は独立なので,

$$P(X_1 + X_2 = i) = \sum_{j=0}^{i} P(X_1 = j)P(X_2 = i - j)$$

$$
\begin{aligned}
&= \sum_{j=0}^{i} e^{-\lambda_1} \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \frac{\lambda_2^{i-j}}{(i-j)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{j=0}^{i} \frac{\lambda_1^j}{j!} \cdot \frac{\lambda_2^{i-j}}{(i-j)!} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{i!} \sum_{j=0}^{i} {}_i\mathrm{C}_j \, \lambda_1^j \lambda_2^{i-j} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^i}{i!}
\end{aligned}
$$

を得る．よって，$X_1 + X_2$ はパラメータ $\lambda_1 + \lambda_2$ のポアソン分布に従う．