

5 重積分

5.5 重積分の応用

問 1

(1) 合成関数の偏微分より, $z_r = \cos \theta z_x + \sin \theta z_y$, $z_\theta = -r \sin \theta z_x + r \cos \theta z_y$ なので, $z_x = \cos \theta z_r - \frac{\sin \theta}{r} z_\theta$, $z_y = \sin \theta z_r + \frac{\cos \theta}{r} z_\theta$. したがって, $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + z_r^2 + \frac{1}{r^2} z_\theta^2}$ であり, 極座標変換より,

$$S = \iint_D \sqrt{1 + z_x^2 + z_y^2} dx dy = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} r \sqrt{1 + z_r^2 + \frac{1}{r^2} z_\theta^2} dr$$

が成り立つ.

(2) $x'(t) \neq 0$ なので, $x = x(t)$ の逆関数が存在する. したがって, 適当な関数 $f(x)$ を用いて, $y = f(x)$ とかける. このとき, $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ なので, 定理 9 で置換積分 $x = x(t)$ を行えば, $S = 2\pi \int_\alpha^\beta |y(t)| \sqrt{x'(t)^2 + y'(t)^2} dt$ が得られる.

問 2 求める曲面積を S とする.

(1) $z = \sqrt{1 - x^2 - y^2}$, $D: x^2 + y^2 \leq 1$ とすると, $z_x = -\frac{x}{\sqrt{1 - x^2 - y^2}}$, $z_y = -\frac{y}{\sqrt{1 - x^2 - y^2}}$ なので, 対称性と極座標変換より,

$$S = 2 \iint_D \frac{dx dy}{\sqrt{1 - x^2 - y^2}} = 2 \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr = 4\pi \left[-\sqrt{1 - r^2} \right]_0^1 = 4\pi.$$

(2) $D: x^2 + y^2 \leq 1$ とすると, $z_x = 2x$, $z_y = 2y$ なので, 極座標変換より,

$$\begin{aligned} S &= \iint_D \sqrt{1 + 4x^2 + 4y^2} dx dy = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 + 4r^2} r dr \\ &= 2\pi \left[\frac{1}{12} (1 + 4r^2)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{6} (5\sqrt{5} - 1). \end{aligned}$$

(3) $z = \sqrt{4 - x^2}$, $D: x^2 + y^2 \leq 4$ とすると, $z_x = -\frac{x}{\sqrt{4 - x^2}}$, $z_y = 0$ なので, 対称性より,

$$S = 2 \iint_D \frac{2}{\sqrt{4 - x^2}} dx dy = 16 \int_0^2 dx \int_0^{\sqrt{4 - x^2}} \frac{dy}{\sqrt{4 - x^2}} = 16 \int_0^2 dx = 32.$$

(4) $z = \sqrt{5x - x^2}$ とすると, $z_x = \frac{5 - 2x}{2\sqrt{5x - x^2}}$, $z_y = 0$ なので, 対称性より,

$$S = 4 \int_0^5 dx \int_0^{\sqrt{25 - 5x}} \frac{5}{2\sqrt{5x - x^2}} dy = 10\sqrt{5} \int_0^5 \frac{dx}{\sqrt{x}} = 10\sqrt{5} [2\sqrt{x}]_0^5 = 100.$$

(5) 球面 $x^2 + y^2 + z^2 = 1$ と $3x^2 + 4y^2 = z + 1$ の交線 $x^2 + y^2 + (3x^2 + 4y^2 - 1)^2 = 1$ は, 極座標 $x = r \cos \theta$, $y = r \sin \theta$ を用いると, $r^2 + (3r^2 + r^2 \sin^2 \theta - 1)^2 = 1$. r について解くと, $r = \frac{\sqrt{5 + 2 \sin^2 \theta}}{3 + \sin^2 \theta}$. ここで, $z = \sqrt{1 - r^2}$ であり, $z_r = -\frac{r}{\sqrt{1 - r^2}}$, $z_\theta = 0$ なので, $\gamma(\theta) = \frac{\sqrt{5 + 2 \sin^2 \theta}}{3 + \sin^2 \theta}$ とおけば,

$$S = \int_0^{2\pi} d\theta \int_0^{\gamma(\theta)} \frac{r}{\sqrt{1 - r^2}} dr = \int_0^{2\pi} \left[-\sqrt{1 - r^2} \right]_0^{\gamma(\theta)} d\theta = \int_0^{2\pi} \left(1 - \frac{2 + \sin^2 \theta}{3 + \sin^2 \theta} \right) d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \frac{d\theta}{3 + \sin^2 \theta} = 4 \int_0^{\frac{\pi}{2}} \frac{1}{4 \tan^2 \theta + 3} \frac{d\theta}{\cos^2 \theta}.$$

ここで, $t = \tan \theta$ とすると, $\frac{dt}{d\theta} = \frac{1}{\cos^2 \theta}$ なので,

$$S = \int_0^{\infty} \frac{dt}{t^2 + \frac{3}{4}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2}{\sqrt{3}} t \right]_0^n = \frac{2}{\sqrt{3}} \lim_{n \rightarrow \infty} \tan^{-1} \frac{2}{\sqrt{3}} n = \frac{\pi}{\sqrt{3}}.$$

(6) $y = \pm \sqrt{4 - x^2} + 5$ なので, $y' = \mp \frac{x}{\sqrt{4 - x^2}}$. したがって,

$$\begin{aligned} S &= 2\pi \int_{-2}^2 \left(\sqrt{4 - x^2} + 5 - \sqrt{4 - x^2} + 5 \right) \frac{2}{\sqrt{4 - x^2}} dx = 40\pi \int_{-2}^2 \frac{dx}{\sqrt{4 - x^2}} \\ &= 40\pi \left[\sin^{-1} \frac{x}{2} \right]_{-2}^2 = 40\pi^2. \end{aligned}$$

(7) $y' = -\sin x$ より,

$$S = 2\pi \int_0^{\pi} |\cos x| \sqrt{1 + \sin^2 x} dx = 4\pi \int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin^2 x} dx.$$

$t = \sin x$ と変換して,

$$S = 4\pi \int_0^1 \sqrt{t^2 + 1} dt = 4\pi \left[\frac{1}{2} \left\{ t\sqrt{t^2 + 1} + \log \left(t + \sqrt{t^2 + 1} \right) \right\} \right]_0^1 = 2\pi \left\{ \sqrt{2} + \log \left(1 + \sqrt{2} \right) \right\}.$$

(8) $y = \pm \sqrt{1 - \frac{x^2}{4}}$ なので, $y' = \mp \frac{x}{2\sqrt{4 - x^2}}$ より,

$$\begin{aligned} S &= \frac{\pi}{2} \int_{-2}^2 \sqrt{16 - 3x^2} dx = \sqrt{3}\pi \int_0^2 \sqrt{\frac{16}{3} - x^2} dx \\ &= \sqrt{3}\pi \left[\frac{1}{2} \left(x\sqrt{\frac{16}{3} - x^2} + \frac{16}{3} \sin^{-1} \frac{\sqrt{3}}{4} x \right) \right]_0^2 = 2\pi \left(1 + \frac{4}{3\sqrt{3}}\pi \right). \end{aligned}$$

(9) $y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ であり, $y' = -x^{-\frac{1}{3}}\sqrt{1 - x^{\frac{2}{3}}}$ なので,

$$S = 2\pi \int_{-1}^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} x^{-\frac{1}{3}} dx = 4\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} x^{-\frac{1}{3}} dx.$$

$x = \sin^3 \theta$ とすると, $\frac{dx}{d\theta} = 3 \sin^2 \theta \cos \theta$ なので,

$$S = 12\pi \int_0^{\frac{\pi}{2}} \sin \theta \cos^4 \theta d\theta = 12\pi \left[-\frac{1}{5} \cos^5 \theta \right]_0^{\frac{\pi}{2}} = \frac{12}{5}\pi.$$

問 3 物体 D の全質量を M , 重心を $G(x_0, y_0, z_0)$, z 軸に関する慣性モーメントを I_z とする.

(1) $M = \iiint_D \rho dx dy dz = 48\rho$. 対称性より, $G(0, 0, 0)$. 対称性より,

$$I_z = \iiint_D (x^2 + y^2) \rho dx dy dz = 8\rho \int_0^1 dx \int_0^2 dy \int_0^3 (x^2 + y^2) dz$$

$$\begin{aligned}
&= 24\rho \int_0^1 dx \int_0^2 (x^2 + y^2) dy = 24\rho \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=2} dx \\
&= 24\rho \int_0^1 \left(2x^2 + \frac{8}{3} \right) dx = 24\rho \left[\frac{2}{3} x^3 + \frac{8}{3} \right]_0^1 = 80\rho.
\end{aligned}$$

$$\begin{aligned}
(2) \quad M &= \iiint_D \rho dx dy dz = \rho \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \\
&= \rho \int_0^1 dx \int_0^{1-x} (1-x-y) dy = \rho \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx \\
&= \frac{1}{2} \rho \int_0^1 (1-x)^2 dx = \frac{1}{2} \rho \left[-\frac{(1-x)^3}{3} \right]_0^1 = \frac{\rho}{6}. \\
x_0 &= \frac{1}{M} \iiint_D x \rho dx dy dz = 6 \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} x dz \\
&= 6 \int_0^1 dx \int_0^{1-x} x(1-x-y) dy = 6 \int_0^1 \left[(x-x^2)y - \frac{x}{2} y^2 \right]_{y=0}^{y=1-x} dx \\
&= 3 \int_0^1 (x-2x^2+x^3) dx = 3 \left[\frac{x^2}{2} - \frac{2}{3} x^3 + \frac{x^4}{4} \right]_0^1 = \frac{1}{4}.
\end{aligned}$$

同様にして, $y_0 = z_0 = \frac{1}{4}$. よって, $G\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

$$\begin{aligned}
I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x^2 + y^2) dz \\
&= \rho \int_0^1 dx \int_0^{1-x} (x^2 - x^3 - x^2 y + (1-x)y^2 - y^3) dy \\
&= \rho \int_0^1 \left[(x^2 - x^3)y - \frac{x^2}{2} y^2 + \frac{1-x}{3} y^3 - \frac{y^4}{4} \right]_{y=0}^{y=1-x} dx \\
&= \frac{\rho}{12} \int_0^1 (7x^4 - 16x^3 + 12x^2 - 4x + 1) dx = \frac{\rho}{12} \left[\frac{7}{5} x^5 - 4x^4 + 4x^3 - 2x^2 + x \right]_0^1 = \frac{\rho}{30}.
\end{aligned}$$

$$\begin{aligned}
(3) \quad M &= \iiint_D \rho dx dy dz = \rho \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} dz = \rho \int_1^2 dx \int_0^{2-x} (2-x-y) dy \\
&= \rho \int_1^2 \left[(2-x)y - \frac{y^2}{2} \right]_{y=0}^{y=2-x} dx = \frac{\rho}{2} \int_1^2 (2-x)^2 dx = \frac{\rho}{2} \left[-\frac{(2-x)^3}{3} \right]_1^2 = \frac{\rho}{6}. \\
x_0 &= \frac{1}{M} \iiint_D x \rho dx dy dz = 6 \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} x dz \\
&= 6 \int_1^2 dx \int_0^{2-x} (2x - x^2 - yx) dy = 6 \int_1^2 \left[(2x-x^2)y - \frac{x}{2} y^2 \right]_{y=0}^{y=2-x} dx \\
&= 3 \int_1^2 (4x - 4x^2 + x^3) dx = 3 \left[2x^2 - \frac{4}{3} x^3 + \frac{x^4}{4} \right]_1^2 = \frac{5}{4}.
\end{aligned}$$

同様にして, $z_0 = \frac{5}{4}$.

$$\begin{aligned}
y_0 &= \frac{1}{M} \iiint_D y \rho dx dy dz = 6 \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} y dz \\
&= 6 \int_1^2 dx \int_0^{2-x} \left\{ (2-x)y - y^2 \right\} dy = 6 \int_1^2 \left[\frac{2-x}{2} y^2 - \frac{y^3}{3} \right]_{y=0}^{y=2-x} dx
\end{aligned}$$

$$= \int_1^2 (2-x)^3 dx = \left[-\frac{(2-x)^4}{4} \right]_1^2 = \frac{1}{4}.$$

よって, $G\left(\frac{5}{4}, \frac{1}{4}, \frac{5}{4}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} (x^2 + y^2) dz \\ &= \rho \int_1^2 dx \int_0^{2-x} \{2x^2 - x^3 - x^2 y + (2-x)y^2 - y^3\} dy \\ &= \rho \int_1^2 \left[(2x^2 - x^3)y - \frac{x^2}{2}y^2 + \frac{2-x}{3}y^3 - \frac{y^4}{4} \right]_{y=0}^{y=2-x} dx \\ &= \frac{\rho}{12} \int_1^2 (7x^4 - 32x^3 + 48x^2 - 32x + 16) dx \\ &= \frac{\rho}{12} \left[\frac{7}{5}x^5 - 8x^4 + 16x^3 - 16x^2 + 16x \right]_1^2 = \frac{17}{60}\rho. \end{aligned}$$

(4) $E: x^2 + y^2 \leq 1, x \geq 0, y \geq 0$ とする.

$$\begin{aligned} M &= \iiint_D \rho dx dy dz = \rho \int_0^1 dz \iint_E dx dy = \frac{\pi}{4}\rho. \\ x_0 &= \frac{1}{M} \iiint_D x \rho dx dy dz = \frac{4}{\pi} \int_0^1 dz \iint_E x dx dy = \frac{4}{\pi} \int_0^1 dy \int_0^{\sqrt{1-y^2}} x dx \\ &= \frac{4}{\pi} \int_0^1 \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \frac{2}{\pi} \int_0^1 (1-y^2) dy = \frac{2}{\pi} \left[y - \frac{y^3}{3} \right]_0^1 = \frac{4}{3\pi}. \end{aligned}$$

同様にして, $y_0 = \frac{4}{3\pi}$. $z_0 = \frac{1}{M} \iiint_D z \rho dx dy dz = \frac{4}{\pi} \int_0^1 dz \iint_E z dx dy = \int_0^1 z dz = \frac{1}{2}$. よって, $G\left(\frac{4}{3\pi}, \frac{4}{3\pi}, \frac{1}{2}\right)$. 極座標変換より,

$$I_z = \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_0^1 dz \iint_E (x^2 + y^2) dx dy = \rho \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 dr = \frac{\pi}{8}\rho.$$

(5) $x = u - 1, v = v + 1, z = w$ により D は $E: u^2 + v^2 \leq 1, 0 \leq w \leq 1$ に写る. $F: u^2 + v^2 \leq 1$ とおく.

$$\begin{aligned} M &= \iiint_D \rho dx dy dz = \rho \iiint_E du dv dw = \rho \int_0^1 dw \iint_F du dv = \pi\rho. \\ x_0 &= \frac{1}{M} \iiint_D (u-1) \rho du dv dw = -1. \text{ 同様にして, } y_0 = 1. \end{aligned}$$

$$z_0 = \frac{1}{M} \iiint_D z \rho dz dy dz = \frac{1}{\pi} \int_0^1 dw \iint_F w du dv = \frac{1}{2}.$$

よって, $G\left(-1, 1, \frac{1}{2}\right)$. 極座標変換より,

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_0^1 dw \iint_F \{(u-1)^2 + (v+1)^2\} du dv \\ &= \rho \int_0^{2\pi} d\theta \int_0^1 \{r^3 - 2r^2(\cos\theta + \sin\theta) + 2r\} dr = \rho \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^3}{3}(\cos\theta + \sin\theta) + r^2 \right]_{r=0}^{r=1} d\theta \end{aligned}$$

$$= \rho \int_0^{2\pi} \left\{ \frac{5}{4} - \frac{1}{3}(\cos \theta + \sin \theta) \right\} d\theta = \frac{5}{2}\pi\rho.$$

(6) 極座標変換より,

$$\begin{aligned} M &= \iiint_D \rho dx dy dz = \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2 \sin \theta dr \\ &= \frac{\pi}{2}\rho \left(\int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) \left(\int_0^1 r^2 dr \right) = \frac{\pi}{2}\rho [-\cos \theta]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^1 = \frac{\pi}{6}\rho. \\ x_0 &= \frac{1}{M} \iiint_D x \rho dx dy dz = \frac{6}{\pi} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 \sin^2 \theta \cos \varphi dr \\ &= \frac{6}{\pi} \left(\int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \right) \left(\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta \right) \left(\int_0^1 r^3 dr \right) \\ &= \frac{3}{\pi} [\sin \varphi]_0^{\frac{\pi}{2}} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^1 = \frac{3}{8}. \end{aligned}$$

同様にして, $y_0 = z_0 = \frac{3}{8}$. よって, $G\left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^4 \sin^3 \theta dr \\ &= \frac{\pi}{2}\rho \left(\int_0^{\frac{\pi}{2}} \frac{3 \sin \theta - \sin 3\theta}{4} d\theta \right) \left(\int_0^1 r^4 dr \right) = \frac{\pi}{8}\rho \left[-3 \cos \theta + \frac{\cos 3\theta}{3} \right]_0^{\frac{\pi}{2}} \left[\frac{r^5}{5} \right]_0^1 = \frac{\pi}{15}\rho. \end{aligned}$$

(7) D は $(-1, -1, 1)$ を中心とする半径 1 の球なので, $G(-1, -1, 1)$. $x = u - 1$, $y = v - 1$, $z = w + 1$ により, D は $E: u^2 + v^2 + z^2 \leq 1$ に写るので, 極座標変換より,

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \iiint_E \{(u-1)^2 + (v-1)^2\} du dv dw \\ &= \rho \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \{r^4 \sin^3 \theta - 2r^3 \sin^2 \theta (\cos \varphi + \sin \varphi) + 2r^2 \sin \theta\} dr \\ &= \rho \int_0^1 dr \int_0^{\pi} [(r^4 \sin^3 \theta + 2r^2 \sin \theta)\varphi - 2r^3 \sin^2 \theta (\sin \varphi - \cos \varphi)]_{\varphi=0}^{\varphi=2\pi} d\theta \\ &= 2\pi\rho \int_0^1 dr \int_0^{\pi} \left(r^4 \frac{3 \sin \theta - \sin 3\theta}{4} + 2r^2 \sin \theta \right) d\theta \\ &= 2\pi\rho \int_0^1 \left[r^4 \left(-3 \frac{\cos \theta}{4} + \frac{\cos 3\theta}{12} \right) - 2r^2 \cos \theta \right]_{\theta=0}^{\theta=\pi} dr \\ &= 2\pi\rho \int_0^1 \left(\frac{4}{3}r^4 + 4r^2 \right) dr = \frac{16}{5}\pi\rho. \end{aligned}$$

(8) $x = r \cos \theta$, $y = r \sin \theta$ とすると,

$$\begin{aligned} M &= \iiint_D \rho dx dy dz = \rho \int_0^1 dz \int_0^{2\pi} d\theta \int_0^{1-z} r dr = \rho \int_0^1 dz \int_0^{2\pi} \frac{(1-z)^2}{2} d\theta \\ &= \pi\rho \int_0^1 (1-z)^2 dz = \pi\rho \left[-\frac{(1-z)^3}{3} \right]_0^1 = \frac{\pi}{3}\rho. \end{aligned}$$

対称性より, $x_0 = y_0 = 0$.

$$z_0 = \frac{1}{M} \iiint_D z \rho dx dy dz = \frac{3}{\pi} \int_0^1 dz \int_0^{2\pi} d\theta \int_0^{1-z} r z dr = \frac{3}{\pi} \int_0^1 dz \int_0^{2\pi} z \frac{(1-z)^2}{2} d\theta$$

$$= 3 \int_0^1 (z - 2z^2 + z^3) dz = 3 \left[\frac{z^2}{2} - \frac{2}{3}z^3 + \frac{z^4}{4} \right]_0^1 = \frac{1}{4}.$$

よって, $G\left(0, 0, \frac{1}{4}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dz dy dx = \rho \int_0^1 dz \int_0^{2\pi} d\theta \int_0^{1-z} r^3 dr \\ &= 2\pi\rho \int_0^1 \frac{(1-z)^4}{4} dz = \frac{\rho}{2} \left[-\frac{(1-z)^5}{5} \right]_0^1 = \frac{\pi}{10}\rho. \end{aligned}$$

(9) $x = 2r \sin \theta \cos \varphi$, $y = \frac{1}{2}r \sin \theta \sin \varphi$, $z = r \cos \theta$ と変換して,

$$\begin{aligned} M &= \iiint_D \rho dx dy dz = \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2 \sin \theta dr \\ &= \pi\rho \left(\int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) \left(\int_0^1 r^2 dr \right) = \pi\rho [-\cos \theta]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^1 = \frac{\pi}{3}\rho. \\ x_0 &= \frac{1}{M} \iiint_D x \rho dx dy dz = \frac{3}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 2r^3 \sin^2 \theta \cos \varphi dr \\ &= \frac{6}{\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi d\varphi \right) \left(\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta \right) \left(\int_0^1 r^3 dr \right) \\ &= \frac{3}{\pi} [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^1 = \frac{3}{4}. \end{aligned}$$

対称性より, $y_0 = 0$.

$$\begin{aligned} z_0 &= \frac{1}{M} \iiint_D z \rho dx dy dz = \frac{3}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 \sin \theta \cos \theta dr \\ &= 3 \left(\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} d\theta \right) \left(\int_0^1 r^3 dr \right) = \frac{3}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^1 = \frac{3}{8}. \end{aligned}$$

よって, $G\left(\frac{3}{4}, 0, \frac{3}{8}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \rho dx dy dz = \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \left(4 \cos^2 \varphi + \frac{1}{4} \sin^2 \varphi \right) r^4 \sin^3 \theta dr \\ &= \rho \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ 2(\cos 2\varphi + 1) + \frac{1 - \cos 2\varphi}{8} \right\} d\varphi \right) \left(\int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) \left(\int_0^1 r^4 dr \right) \\ &= \rho \left[\sin 2\varphi + \frac{17}{8}\varphi - \frac{\sin 2\varphi}{16} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [-\cos \theta]_0^{\frac{\pi}{2}} \left[\frac{r^5}{5} \right]_0^1 = \frac{17}{80}\pi\rho. \end{aligned}$$