5　重積分

5.1　2 重積分

問 1　(1)　$$\iint_D x^2y^3dxdy = \left(\int_{-1}^1 x^2dx\right)\left(\int_0^1 y^3dy\right) = \left[\frac{x^3}{3}\right]_{-1}^1 \cdot \left[\frac{y^4}{4}\right]_0^1 = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}.$$

(2)　$$\iint_D (\cos x)ydxdy = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos xdx\right)\left(\int_0^1 ydy\right) = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \left[\frac{x^2}{2}\right]_0^1 = 1.$$

(3)　$$\iint_D (x+y)dxdy = \int_{-1}^1 \left\{\int_0^1 (x+y)dx\right\}dy = \int_{-1}^1 \left[\frac{(x+y)^2}{2}\right]_{x=0}^{x=1} dy$$
$$= \frac{1}{2}\int_{-1}^1 \left\{(y+1)^2 - y^2\right\}dy = \frac{1}{2}\int_{-1}^1 (2y+1)dy = \int_0^1 dy = 1.$$

(4)　$$\iint_D \sin(x+y)dxdy = \int_0^{\frac{\pi}{4}} \left\{\int_0^{\frac{3\pi}{4}} \sin(x+y)dx\right\}dy = \int_0^{\frac{\pi}{4}} [-\cos(x+y)]_{x=0}^{x=\frac{3\pi}{4}} dy$$
$$= \int_0^{\frac{\pi}{4}} \left\{-\cos\left(y+\frac{3\pi}{4}\right) + \cos y\right\}dy = \left[-\sin\left(y+\frac{3\pi}{4}\right) + \sin y\right]_0^{\frac{\pi}{4}} = \sqrt{2}.$$

問 2

(1)　$D\colon 0 \leqq x \leqq 1,\ x^2 \leqq y \leqq x$ とおく．求める面積は,

$$\iint_D dxdy = \int_0^1 dx\int_{x^2}^x dy = \int_0^1 (x-x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

(2)　$D\colon -1 \leqq x \leqq 1,\ x^2 \leqq y \leqq -x^2+2$ とおく．求める面積は

$$\iint_D dxdy = \int_{-1}^1 dx\int_{x^2}^{-x^2+2} dy = \int_{-1}^1 (-2x^2+2)dx = 4\int_0^1 (-x^2+1)dx = 4\left[-\frac{x^3}{3}+x\right]_0^1 = \frac{8}{3}.$$

(3)　$D\colon 0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \sin^2 x$ とおく．$\sin^2 x = \dfrac{1-\cos(2x)}{2}$ なので，対称性より，求める面積は

$$2\iint_D dxdy = \int_0^\pi \{1-\cos(2x)\}dx = \left[x - \frac{\sin(2x)}{2}\right]_0^\pi = \pi.$$

(4)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq (1-\sqrt{x})^2$ とおくと，対称性より，求める面積は

$$4\iint_D dxdy = 4\int_0^1 \left(1-\sqrt{x}\right)^2 dx = 4\int_0^1 \left(1-2\sqrt{x}+x\right)dx = 4\left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3}.$$

問 3　(1)　$$\iint_D xy^2dxdy = \int_{-1}^1 dx\int_{-1}^{-x} xy^2dy = \int_{-1}^1 \left[x\frac{y^3}{3}\right]_{y=-1}^{y=-x} dx = \int_{-1}^1 \frac{-x^4+x}{3}dx$$
$$= -\frac{2}{3}\int_0^1 x^4dx = -\frac{2}{3}\left[\frac{x^5}{5}\right]_0^1 = -\frac{2}{15}.$$

(2)　$$\iint_D \sin(x+y)dxdy = \int_0^\pi dx\int_0^{-x+\pi} \sin(x+y)dy$$
$$= \int_0^\pi [-\cos(x+y)]_{y=0}^{y=-x+\pi} dx = \int_0^\pi (1+\cos x)dx = [x+\sin x]_0^\pi = \pi.$$

(3)　$$\iint_D \sqrt{x^4+1}dxdy = \int_0^1 dx\int_0^{x^3} \sqrt{x^4+1}dy = \int_0^1 x^3\sqrt{x^4+1}dx$$
$$= \left[\frac{1}{6}(x^4+1)^{\frac{3}{2}}\right]_0^1 = \frac{2\sqrt{2}-1}{6}.$$

(4) $$\iint_D e^{\frac{x}{y}}dxdy = \int_0^1 dy\int_0^{y^2} e^{\frac{x}{y}}dx = \int_0^1 \left[ye^{\frac{x}{y}}\right]_{x=0}^{x=y^2} dy = \int_0^1 y\left(e^y-1\right)dy$$
$$= \left[ye^y\right]_0^1 - \int_0^1 e^y dy - \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

(5) $$\iint_D |y|dxdy = \int_{-1}^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |y|dx = 2\int_{-1}^1 |y|\sqrt{1-y^2}dy$$
$$= 4\int_0^1 y\sqrt{1-y^2}dy = 4\left[-\frac{1}{3}(1-y^2)^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}.$$

(6) $$\iint_D xy^2dxdy = \int_0^1 dx\int_x^{\sqrt{x}} xy^2dy + \int_0^1 dx\int_{-\sqrt{x}}^{-x} xy^2dy = 2\int_0^1 x\left[\frac{y^3}{3}\right]_{y=x}^{y=\sqrt{x}} dx$$
$$= 2\int_0^1 \frac{x^{\frac{5}{2}}-x^4}{3}dx = \frac{2}{3}\left[\frac{2}{7}x^{\frac{7}{2}} - \frac{x^5}{5}\right]_0^1 = \frac{2}{3}\left(\frac{2}{7}-\frac{1}{5}\right) = \frac{2}{35}.$$

5.2　変数変換

問 1

(1)　$u=x+2y$，$v=3x-y$ とおくと $x=\dfrac{u+2v}{7}$，$y=\dfrac{3u-v}{7}$ なので，$\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{7} & \frac{2}{7}\\ \frac{3}{7} & -\frac{1}{7}\end{pmatrix}$ $=-\dfrac{1}{7}$．よって，

$$\begin{aligned}\iint_D \cos(x+2y)\sin(3x-y)dxdy &= \frac{1}{7}\left(\int_{\frac{\pi}{2}}^{\pi}\cos u du\right)\left(\int_{-\frac{\pi}{2}}^{0}\sin v dv\right)\\ &= \frac{1}{7}\left[\sin u\right]_{\frac{\pi}{2}}^{\pi}\cdot\left[-\cos v\right]_{-\frac{\pi}{2}}^{0}=\frac{1}{7}.\end{aligned}$$

(2)　$x=r\cos\theta$，$y=r\sin\theta$ と変換して，

$$\iint_D \sin(x^2+y^2)dxdy = 2\pi\int_0^{\sqrt{\pi}}\sin(r^2)rdr = 2\pi\left[-\frac{\cos(r^2)}{2}\right]_0^{\sqrt{\pi}} = 2\pi.$$

(3)　$u=x+y$，$v=x-y$ とおくと $x=\dfrac{u+v}{2}$，$y=\dfrac{u-v}{2}$ なので，$\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=$ $-\dfrac{1}{2}$．よって，

$$\begin{aligned}&\iint_D (x+y)^2|\cos\pi(x-y)|dxdy = \frac{1}{2}\left(\int_{-1}^{1}u^2du\right)\left(\int_{-1}^{1}|\cos\pi v|dv\right)\\ &=4\left(\int_0^1 u^2du\right)\left(\int_0^{\frac{1}{2}}\cos\pi v dv\right)=4\left[\frac{u^3}{3}\right]_0^1\cdot\left[\frac{\sin\pi v}{\pi}\right]_0^{\frac{1}{2}}=\frac{4}{3\pi}.\end{aligned}$$

(4)　$u=x+y$，$v=x-y$ とおくと $x=\dfrac{u+v}{2}$，$y=\dfrac{u-v}{2}$ なので $\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=$ $-\dfrac{1}{2}$．よって，

$$\iint_D (x^2-y^2)^2dxdy = \frac{1}{2}\left(\int_{-2}^{2}u^2du\right)\left(\int_{-2}^{2}v^2dv\right) = 2\left(\int_0^2 u^2du\right)^2 = 2\left(\left[\frac{u^3}{3}\right]_0^2\right)^2 = \frac{128}{9}.$$

(5)　$x=3r\cos\theta$，$y=2r\sin\theta$ とおくと $\dfrac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}3\cos\theta & -3r\sin\theta\\ 2\sin\theta & 2r\cos\theta\end{pmatrix}=6r$ なので，

$$\begin{aligned}&\iint_D x^2y^2dxdy = 6\int_0^{2\pi}d\theta\int_0^1 r^5\cos^2\theta\sin^2\theta dr = 6\left(\int_0^{2\pi}\frac{\sin^2 2\theta}{4}d\theta\right)\left(\int_0^1 r^5dr\right)\\ &=6\int_0^{2\pi}\frac{1-\cos4\theta}{8}d\theta\left[\frac{r^6}{6}\right]_0^1=\frac{1}{8}\left[\theta-\frac{\sin4\theta}{4}\right]_0^{2\pi}=\frac{\pi}{4}.\end{aligned}$$

(6)　$x=r\cos\theta$，$y=r\sin\theta$ とおくと，E: $0\leqq r\leqq\sqrt{\cos2\theta}$, $-\dfrac{\pi}{4}\leqq\theta\leqq\dfrac{\pi}{4}$ は D に写るので，

$$\begin{aligned}&\iint_D \frac{dxdy}{(1+x^2+y^2)^2} = 2\int_0^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{\cos2\theta}}\frac{r}{(1+r^2)^2}dr\\ &=2\int_0^{\frac{\pi}{4}}\left[-\frac{1}{2}\frac{1}{1+r^2}\right]_0^{\sqrt{\cos2\theta}}d\theta=\int_0^{\frac{\pi}{4}}\frac{\cos2\theta}{1+\cos2\theta}d\theta\\ &=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(1-\tan^2\theta\right)d\theta=\frac{1}{2}\left(\frac{\pi}{4}-\left[\tan\theta-\theta\right]_0^{\frac{\pi}{4}}\right)=\frac{\pi-2}{4}.\end{aligned}$$

5.3　広義 2 重積分

問 1 $J = \iint_D e^{-x^2-y^2} x^{2p-1} y^{2q-1} dxdy$ とおく．近似列 $D_n = \{(x,y) \mid x^2+y^2 \leqq n^2,\ x \geqq 0,\ y \geqq 0\}$ では，$s = r^2$ とすると $\dfrac{dr}{ds} = \dfrac{1}{2\sqrt{s}}$ なので，

$$
\begin{aligned}
J &= \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} d\theta \int_0^n e^{-r^2} r^{2(p+q)-1} \cos^{2p-1}\theta \sin^{2q-1}\theta dr \\
&= \frac{1}{2}\left(\int_0^{\infty} e^{-s} s^{-p-q-1} ds\right)\left(\int_0^{\frac{\pi}{2}} \cos^{2p-1}\theta \sin^{2q-1} d\theta\right) = \frac{1}{4}\Gamma(p+q)B(p,q).
\end{aligned}
$$

一方，$E_n = \{(x,y) \mid 0 \leqq x \leqq n,\ 0 \leqq y \leqq n\}$ では，$u = x^2$，$v = y^2$ とすると $\dfrac{dx}{du} = \dfrac{1}{2\sqrt{u}}$，$\dfrac{dy}{dv} = \dfrac{1}{2\sqrt{v}}$ なので，

$$
J = \lim_{n\to\infty}\left(\int_0^n e^{-x^2} x^{2p-1} dx\right)\left(\int_0^n e^{-y^2} y^{2q-1} dy\right) = \frac{1}{4}\Gamma(p)\Gamma(q).
$$

よって，$B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ である．

問 2

(1)　$D_n\colon 0 \leqq x \leqq n,\ 0 \leqq y \leqq n$ とすると，

$$
\begin{aligned}
&\iint_D \frac{dxdy}{(x+y+1)^3} = \lim_{n\to\infty}\iint_{D_n} \frac{dxdy}{(x+y+1)^3} = \lim_{n\to\infty}\int_0^n dx \int_0^n \frac{dy}{(x+y+1)^3} \\
&= \lim_{n\to\infty}\int_0^n \left[-\frac{1}{2}(x+y+1)^{-2}\right]_{y=0}^{y=n} dx \\
&= -\frac{1}{2}\lim_{n\to\infty}\int_0^n \left\{(x+n+1)^{-2} - (x+1)^{-2}\right\} dx \\
&= -\frac{1}{2}\lim_{n\to\infty}\left[-\left\{(x+n+1)^{-1} - (x+1)^{-1}\right\}\right]_0^n = \frac{1}{2}.
\end{aligned}
$$

(2)　$D_n\colon x^2+y^2 \leqq n^2$ とし，$x = r\cos\theta$，$y = r\sin\theta$ とおくと，

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} \frac{dxdy}{(x^2+y^2+1)^{\frac{3}{2}}} = \lim_{n\to\infty}\iint_{D_n} \frac{dxdy}{(x^2+y^2+1)^{\frac{3}{2}}} \\
&= \lim_{n\to\infty}\int_0^n dr \int_0^{2\pi} \frac{r}{(r^2+1)^{\frac{3}{2}}} d\theta = 2\pi \lim_{n\to\infty}\int_0^n \left[-\left(r^2+1\right)^{-\frac{1}{2}}\right]_0^n = 2\pi.
\end{aligned}
$$

(3)　$D_n\colon x^2+y^2 \leqq 4 - \dfrac{1}{n}$ とし，$x = r\cos\theta$，$y = r\sin\theta$ とおくと，

$$
\begin{aligned}
&\iint_D \frac{dxdy}{\sqrt{4-x^2-y^2}} = \lim_{n\to\infty}\iint_{D_n} \frac{dxdy}{\sqrt{4-x^2-y^2}} \\
&= \lim_{n\to\infty}\int_0^{\sqrt{4-\frac{1}{n}}} dr \int_0^{2\pi} \frac{r}{\sqrt{4-r^2}} d\theta = 2\pi \lim_{n\to\infty}\left[-\sqrt{4-r^2}\right]_0^{\sqrt{4-\frac{1}{n}}} = 4\pi.
\end{aligned}
$$

(4)　$D_n\colon x^2+y^2 \leqq n^2,\ x \geqq 0,\ y \geqq 0$ とし，$x = r\cos\theta$，$y = r\sin\theta$ とおくと，

$$
\iint_D e^{-x^2-y^2} dxdy = \lim_{n\to\infty}\iint_{D_n} e^{-x^2-y^2} dxdy = \lim_{n\to\infty}\int_0^n dr \int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta
$$

$$= \frac{\pi}{2}\lim_{n\to\infty}\left[-\frac{1}{2}e^{-r^2}\right]_0^n = \frac{\pi}{4}.$$

(5)　$D_n\colon 1 \leqq x \leqq n,\ 1 \leqq y \leqq n$ とすると，

$$\iint_D \frac{dxdy}{(x+y)^4} = \lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{(x+y)^4} = \lim_{n\to\infty}\int_1^n dx\int_1^n \frac{dy}{(x+y)^4}$$
$$= \lim_{n\to\infty}\int_1^n \left[-\frac{1}{3}(x+y)^{-3}\right]_{y=1}^{y=n} dx = \frac{1}{3}\lim_{n\to\infty}\int_1^n \left\{(x+1)^{-3}-(x+n)^{-3}\right\}dx$$
$$= \frac{1}{3}\lim_{n\to\infty}\left[-\frac{1}{2}\left\{(x+1)^{-2}-(x+n)^{-2}\right\}\right]_1^n = \frac{1}{24}.$$

(6)　$D_n\colon \frac{1}{n} \leqq y \leqq 1,\ \frac{1}{n} \leqq x \leqq y$ とすると，

$$\iint_D \frac{x^2}{x^3+y^3}dxdy = \lim_{n\to\infty}\iint_{D_n}\frac{x^2}{\sqrt{x^3+y^3}}dxdy = \lim_{n\to\infty}\int_{\frac{1}{n}}^1 dy\int_{\frac{1}{n}}^y \frac{x^2}{x^3+y^3}dx$$
$$= \lim_{n\to\infty}\int_{\frac{1}{n}}^1\left[\frac{1}{3}\log(x^3+y^3)\right]_{x=\frac{1}{n}}^{x=y} dy = \frac{1}{3}\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \log\frac{2y^3}{\frac{1}{n^3}+y^3}dy.$$

ここで，$y \geq \frac{1}{n}$ のとき，

$$\left|\log\frac{2y^3}{\frac{1}{n^3}+y^3}-\log 2\right| = \left|\log\frac{y^3}{\frac{1}{n^3}+y^3}\right| = \left|\log\left(1-\frac{\frac{1}{n^3}}{\frac{1}{n^3}+y^3}\right)\right| \leq \frac{\frac{1}{n^3}}{\frac{1}{n^3}+y^3} \leq \frac{1}{1+n^2y^2}.$$

また，

$$\int_{\frac{1}{n}}^1 \frac{dy}{1+n^2y^2} = \left[\frac{1}{n}\tan^{-1}(ny)\right]_{\frac{1}{n}}^1 = \frac{\tan^{-1}n-\frac{\pi}{4}}{n} \to 0 \quad (n\to\infty)$$

なので，$\displaystyle\iint_D \frac{x^2}{x^3+y^3}dxdy = \frac{1}{3}\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \log 2dy = \frac{\log 2}{3}.$

問 3　求める極限を I とおく．

(1)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y)\mid 0<x\leqq 1,\ 0<y\leqq 1\}$ なので，D の近似列である．$\displaystyle\frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{1}{x^2+y^2}+\frac{2x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)$ なので，

$$I = \lim_{n\to\infty}\int_{\frac{1}{n}}^1\left[-\frac{x}{x^2+y^2}\right]_{\frac{1}{n}}^1 dy = \lim_{n\to\infty}\int_{\frac{1}{n}}^1\left(-\frac{1}{1+y^2}+\frac{1}{n}\frac{1}{y^2+\frac{1}{n^2}}\right)dy$$
$$= \lim_{n\to\infty}\left[-\tan^{-1}y+\tan^{-1}ny\right]_{\frac{1}{n}}^1 = 0.$$

(2)　D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y)\mid 0<x\leqq 1,\ 0<y\leqq 1\}$ なので，D の近似列である．

$$I = \lim_{n\to\infty}\int_{\frac{\sqrt{3}}{n}}^1\left[-\frac{x}{x^2+y^2}\right]_{\frac{1}{n}}^1 dy = \lim_{n\to\infty}\int_{\frac{\sqrt{3}}{n}}^1\left(-\frac{1}{1+y^2}+\frac{1}{n}\frac{1}{y^2+\frac{1}{n^2}}\right)dy$$
$$= \lim_{n\to\infty}\left[-\tan^{-1}y+\tan^{-1}ny\right]_{\frac{\sqrt{3}}{n}}^1 = -\frac{\pi}{12}.$$

(3)

$$D_n = \left\{ (x,y) \middle| 0 \leqq x \leqq 1,\, x\tan\frac{1}{n} \leqq y \leqq x\tan\left(\frac{\pi}{2} - \frac{2}{n}\right),\, x^2+y^2 \geqq e^{-2n^2} \right\}$$

と表せる．故に，D_n は縦線集合で，$D_n \subset D_{n+1}$，$\displaystyle\bigcup_{n=1}^{\infty} D_n = \{(x,y) \mid 0 < x \leqq 1,\, 0 < y \leqq 1\}$ なので，D の近似列である．極座標変換より，

$$\begin{aligned} I &= \lim_{n\to\infty} \left(\int_{\frac{1}{n}}^{\frac{\pi}{4}} d\theta \int_{e^{-n^2}}^{\frac{1}{\cos\theta}} \frac{\cos 2\theta}{r} dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{2}{n}} d\theta \int_{e^{-n^2}}^{\frac{1}{\sin\theta}} \frac{\cos 2\theta}{r} dr \right) \\ &= \lim_{n\to\infty} \left\{ \int_{\frac{1}{n}}^{\frac{\pi}{4}} \cos 2\theta \left(\log\frac{1}{\cos\theta} + n^2 \right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{2}{n}} \cos 2\theta \left(\log\frac{1}{\sin\theta} + n^2 \right) d\theta \right\}. \end{aligned}$$

ここで，$\theta' = -\theta + \dfrac{\pi}{2}$ として，

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{2}{n}} \cos 2\theta \left(\log\frac{1}{\sin\theta} + n^2 \right) d\theta = -\int_{\frac{2}{n}}^{\frac{\pi}{4}} \cos 2\theta' \left(\log\frac{1}{\cos\theta'} + n^2 \right) d\theta'$$

より，

$$\begin{aligned} I &= \lim_{n\to\infty} \int_{\frac{1}{n}}^{\frac{2}{n}} \cos 2\theta \left(\log\frac{1}{\cos\theta} + n^2 \right) d\theta = \lim_{n\to\infty} n^2 \int_{\frac{1}{n}}^{\frac{2}{n}} \cos 2\theta d\theta \\ &= \lim_{n\to\infty} n^2 \left[\frac{\sin(2\theta)}{2} \right]_{\frac{1}{n}}^{\frac{2}{n}} = \lim_{n\to\infty} n^2 \frac{\sin\frac{4}{n} - \sin\frac{2}{n}}{2} = \lim_{n\to\infty} \left(n^2 \cos\frac{3}{n} \sin\frac{1}{n} \right) = \infty. \end{aligned}$$

問 4

(1)　$x = r\cos\theta$，$y = r\sin\theta$ とし，$D_n\colon x^2+y^2 \leqq n^2$ とすると，

$$\iint_{\mathbb{R}^2} \frac{dxdy}{1+(x^2+y^2)^2} = 2\pi \lim_{n\to\infty} \int_0^n \frac{r}{1+r^4} dr = 2\pi \lim_{n\to\infty} \left[\frac{1}{2}\tan^{-1} r^2 \right]_0^n = \frac{\pi^2}{2}.$$

(2)　$D_n\colon 0 \leqq x \leqq n,\, 0 \leqq y \leqq n$ とすると，

$$\begin{aligned} \iint_D ye^{-xy} dxdy &= \lim_{n\to\infty} \int_0^n dy \int_0^n ye^{-xy} dx = \lim_{n\to\infty} \int_0^n \left[-e^{-xy} \right]_{x=0}^{x=n} dy \\ &= \lim_{n\to\infty} \int_0^n \left(1 - e^{-ny} \right) dy = \lim_{n\to\infty} \left[y + \frac{1}{n} e^{-ny} \right]_0^n = \infty \end{aligned}$$

なので，発散する．

(3)　$x = r\cos\theta$，$y = r\sin\theta$ とし，$D_n = \left\{ (r\cos\theta, r\sin\theta) \middle| \dfrac{1}{n^2} \leqq r \leqq 1,\, \dfrac{1}{n} \leqq \theta \leqq 2\pi \right\}$ とすると，極座標変換により，

$$\begin{aligned} \iint_D \frac{x}{(x^2+y^2)^2} dxdy &= \lim_{n\to\infty} \int_{\frac{1}{n^2}}^1 dr \int_{\frac{1}{n}}^{2\pi} \frac{\cos\theta}{r^2} d\theta = 4 \lim_{n\to\infty} \left[-\frac{1}{r} \right]_{\frac{1}{n^2}}^1 \cdot [\sin\theta]_{\frac{1}{n}}^{2\pi} \\ &= 4 \lim_{n\to\infty} (n^2-1) \left(-\sin\frac{1}{n} \right) = -\infty \end{aligned}$$

より，発散する．

(4)　$x = r\cos\theta$，$y = r\sin\theta$ とし，$D_n\colon 1 \leqq x^2+y^2 \leqq n^2$ とすると，

$$\iint_D \frac{|x|}{(x^2+y^2)^2} dxdy = \lim_{n\to\infty} \int_1^n dr \int_0^{2\pi} \frac{|\cos\theta|}{r^2} d\theta = 4 \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{dr}{r^2} = 4 \lim_{n\to\infty} \left[-\frac{1}{r} \right]_1^n = 4$$

より，収束する．また，

$$\iint_D \frac{x}{(x^2+y^2)^2}dxdy = \lim_{n\to\infty}\int_1^n dr\int_0^{2\pi}\frac{\cos\theta}{r^2}d\theta = 0.$$

(5)　D_n: $\dfrac{1}{n} \leqq x+y \leqq 2n$, $-n \leqq x+y \leqq -\dfrac{1}{n}$, $\dfrac{1}{n} \leqq x-y \leqq n^2$, $-n \leqq x-y \leqq -\dfrac{1}{n}$ とする．$u=x+y$, $v=x-y$ とすると，$\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$ なので，

$$\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{x^2-y^2} = \frac{1}{2}\lim_{n\to\infty}\left(\int_n^{2n}\frac{du}{u}\right)\left(\int_n^{n^2}\frac{dv}{v}\right) = \frac{1}{2}\lim_{n\to\infty}(\log 2)\log n = \infty$$

より，発散する．

(6)　$x=\dfrac{u+v}{2}$，$y=\dfrac{u-v}{2}$ により，E_n: $1 \leqq u \leqq n$, $-u \leqq v \leqq u-2$ は D_n: $x \geqq 0$, $y \geqq 1$, $x+y \leqq n$ に写る．また，$\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，

$$\begin{aligned}
&\iint_D \left|\frac{x-y}{(x+y)^5}\right| dxdy = \frac{1}{2}\lim_{n\to\infty}\int_1^n du\int_{-u}^{u-2}\frac{|v|}{u^5}dv \\
&= \frac{1}{2}\int_1^2 du\int_{-u}^{u-2}\frac{-v}{u^5}dv + \frac{1}{2}\lim_{n\to\infty}\int_2^n\left(\int_{-u}^0\frac{-v}{u^5}dv + \int_0^{u-2}\frac{v}{u^5}dv\right)du \\
&= \frac{1}{2}\int_1^2\frac{-(u-2)^2+u^2}{2u^5}du + \frac{1}{2}\lim_{n\to\infty}\int_2^n\left(\frac{1}{2u^3}+\frac{(u-2)^2}{2u^5}\right)du \\
&= \frac{1}{2}\int_1^2\left(\frac{2}{u^4}+\frac{4}{u^5}\right)du + \frac{1}{2}\lim_{n\to\infty}\int_2^n\left(\frac{1}{u^3}-\frac{1}{u^4}+\frac{2}{u^5}\right)du \\
&= \frac{1}{2}\left[-\frac{2}{3u^3}-\frac{1}{u^4}\right]_1^2 + \frac{1}{2}\lim_{n\to\infty}\left[-\frac{1}{2u^2}+\frac{1}{3u^3}-\frac{1}{u^4}\right]_2^n du < \infty
\end{aligned}$$

より，収束する．また，

$$\begin{aligned}
&\iint_D \frac{x-y}{(x+y)^5}dxdy = \frac{1}{2}\lim_{n\to\infty}\int_1^n du\int_{-u}^{u-2}\frac{v}{u^5}dv = \frac{1}{2}\lim_{n\to\infty}\int_1^n\left[\frac{v^2}{2}\right]_{v=-u}^{v=u-2}\frac{1}{u^5}du \\
&= \lim_{n\to\infty}\int_1^n\left(-\frac{1}{u^4}+\frac{1}{u^5}\right)du = \lim_{n\to\infty}\left[\frac{1}{3u^3}-\frac{1}{4u^4}\right]_1^n \\
&= -\frac{1}{3}+\frac{1}{4} = -\frac{1}{12}.
\end{aligned}$$

5.4　3 重積分

問 1

(1)　$U\colon 0 \leqq x \leqq 1,\, 0 \leqq y \leqq 1-x,\, 0 \leqq z \leqq 1-x-y$ とする．求める体積は，対称性より，

$$\begin{aligned}8\iiint_U dxdydz &= 8\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} dz \\ &= 8\int_0^1 dx\int_0^{1-x}(1-x-y)dy = 4\int_0^1 (1-x)^2 dx = \frac{4}{3}.\end{aligned}$$

(2)　$U\colon x^2+y^2 \leqq 1,\, 0 \leqq z \leqq \sqrt{1-x^2-y^2}$ とし，$D\colon x^2+y^2 \leqq 1$ とおく．求める体積は，対称性より，

$$\begin{aligned}2\iiint_U dxdydz &= 2\iint_D dxdy\int_0^{\sqrt{1-x^2-y^2}} dz = 2\iint_D \sqrt{1-x^2-y^2}dxdy \\ &= 4\pi\int_0^1 \sqrt{1-r^2}rdr = 4\pi\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}\pi.\end{aligned}$$

(3)　$U\colon 0 \leqq x \leqq 2,\, 0 \leqq y \leqq \sqrt{4-x^2},\, 0 \leqq z \leqq \sqrt{4-x^2}$ とする．求める体積は，対称性より，

$$8\iiint_U dxdydz = 8\int_0^2 dx\int_0^{\sqrt{4-x^2}} dy\int_0^{\sqrt{4-x^2}} dz = 8\int_0^2 \left(4-x^2\right) dx = 8\left[4x-\frac{x^3}{3}\right]_0^2 = \frac{128}{3}.$$

(4)　$U\colon x^2+y^2 \leqq 1,\, x \leqq z \leqq 2x$ とし，$D\colon x^2+y^2 \leqq 1,\, x \geqq 0$ とおく．求める体積は，

$$\iiint_U dxdydz = \iint_D xdxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^1 r^2\cos\theta dr = [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\left[\frac{r^3}{3}\right]_0^1 = \frac{2}{3}.$$

問 2

(1)　$\displaystyle\iiint_U xydxdydz = \left(\int_0^1 xdx\right)\left(\int_{-1}^0 ydy\right)\left(\int_0^2 dz\right) = \left[\frac{x^2}{2}\right]_0^1\cdot\left[\frac{y^2}{2}\right]_{-1}^0\cdot 2 = -\frac{1}{2}.$

(2)　対称性より，

$$\begin{aligned}\iiint_U (x+y+z)dxdydz &= 3\iiint_U xdxdydz = 3\left(\int_0^1 xdx\right)\left(\int_0^1 dy\right)\left(\int_0^1 dz\right) \\ &= 3\cdot\frac{1}{2}\cdot 1\cdot 1 = \frac{3}{2}.\end{aligned}$$

(3)　$U\colon 0 \leqq x \leqq 2,\, 0 \leqq y \leqq 2-x,\, 0 \leqq z \leqq 2-x-y$ であり，対称性より，

$$\begin{aligned}&\iiint_U (x+y+z)dxdydz = 3\iiint_U xdxdydz = 3\int_0^2 dx\int_0^{2-x} dy\int_0^{2-x-y} xdz \\ &= 3\int_0^2 dx\int_0^{2-x} x(2-x-y)dy = 3\int_0^2 \left[-x\frac{(2-x-y)^2}{2}\right]_{y=0}^{y=2-x} dx \\ &= \frac{3}{2}\int_0^2 x(2-x)^2dx = \frac{3}{2}\left(\left[-x\frac{(2-x)^3}{3}\right]_0^2 + \int_0^2 \frac{(2-x)^3}{3}dx\right) \\ &= \frac{1}{2}\left[-\frac{(2-x)^4}{4}\right]_0^2 = 2.\end{aligned}$$

(4)　$-1 \leqq z \leqq 1$ に対して $D(z)\colon x^2+y^2 \leqq 1-z^2$ とすれば,

$$\iiint_U (x^2+y^2)dxdydz = \int_{-1}^{1} dz \int_{D(z)} (x^2+y^2)dxdy = \int_{-1}^{1} dz \int_0^{\sqrt{1-z^2}} dr \int_0^{2\pi} r^3 d\theta$$
$$= 2\pi \int_{-1}^{1} \left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{1-z^2}} dz = \frac{\pi}{2}\int_{-1}^{1}(1-z^2)^2 dz$$
$$= \pi \int_0^1 (1-2z^2+z^4)dz = \pi\left[z-\frac{2}{3}z^3+\frac{z^5}{5}\right]_0^1 = \frac{8}{15}\pi.$$

(5)　$U\colon 0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \pi - x,\ 0 \leqq z \leqq \pi - x - y$ より,

$$\iiint_U \cos(x+y+z)dxdydz = \int_0^{\pi} dx \int_0^{\pi-x} dy \int_0^{\pi-x-y} \cos(x+y+z)dz$$
$$= \int_0^{\pi} dx \int_0^{\pi-x} [\sin(x+y+z)]_{z=0}^{z=\pi-x-y}\, dy = -\int_0^{\pi} dx \int_0^{\pi-x} \sin(x+y)dy$$
$$= \int_0^{\pi} [\cos(x+y)]_{y=0}^{y=\pi-x}\, dx = \int_0^{\pi} (-1-\cos x)\, dx = [-x-\sin x]_0^{\pi} = -\pi.$$

(6) $U\colon 0 \leqq x \leqq 1,\ x \leqq y \leqq 1,\ y \leqq z \leqq 1$ より,

$$\iiint_U \frac{dxdydz}{\sqrt{x+y+z}} = \int_0^1 dx \int_x^1 dy \int_y^1 \frac{1}{\sqrt{x+y+z}}dz$$
$$= \int_0^1 dx \int_x^1 \left[2(x+y+z)^{\frac{1}{2}}\right]_{z=y}^{z=1} dy$$
$$= 2\int_0^1 dx \int_x^1 \left\{(x+y+1)^{\frac{1}{2}} - (x+2y)^{\frac{1}{2}}\right\} dy$$
$$= 2\int_0^1 \left[\frac{2}{3}(x+y+1)^{\frac{3}{2}} - \frac{1}{3}(x+2y)^{\frac{3}{2}}\right]_{y=x}^{y=1} dx$$
$$= \frac{2}{3}\int_0^1 \left\{(x+2)^{\frac{3}{2}} - 2(2x+1)^{\frac{3}{2}} + (3x)^{\frac{3}{2}}\right\} dx$$
$$= \frac{2}{3}\left[\frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{2}{5}(2x+1)^{\frac{5}{2}} + \frac{2}{15}(3x)^{\frac{5}{2}}\right]_0^1$$
$$= \frac{4}{15}\left(3^{\frac{5}{2}} - 3^{\frac{5}{2}} + \frac{1}{3}3^{\frac{5}{2}} - 2^{\frac{5}{2}} + 1\right)$$
$$= \frac{4}{15}\left(3\sqrt{3} - 4\sqrt{2} + 1\right).$$

問 3

(1)　円柱座標変換より,

$$\iiint_U y^2 dxdydz = \int_0^1 dr \int_0^{2\pi} d\theta \int_{r^2}^1 r^3 \sin^2\theta dz$$
$$= \left\{\int_0^1 (r^3 - r^5)dr\right\}\left(\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta d\theta\right) = \left(\frac{1}{4}-\frac{1}{6}\right)\pi = \frac{\pi}{12}.$$

(2)　極座標変換より,

$$\iiint_U (x^2+y^2+z^2)dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 r^4 \sin\theta dr$$

$$= \left(\int_0^1 r^4 dr\right)\left(\int_0^\pi \sin\theta d\theta\right)\left(\int_0^{2\pi} d\varphi\right) = 2\pi\left[\frac{r^5}{5}\right]_0^1 \cdot [-\cos\theta]_0^\pi = \frac{4}{5}\pi.$$

(3)　$x = u - v + w$，$y = v$，$z = v - w$ により，$W\colon -\pi \leqq u \leqq 0, 0 \leqq v \leqq 1, 0 \leqq w \leqq \dfrac{\pi}{2}$ は U に写り，$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = -1$ なので，

$$\iiint_U \sin(x+z)\cos(y-z)dxdydz = \int_{-\pi}^0 du \int_0^1 dv \int_0^{\frac{\pi}{2}} \sin u \cos w dw$$
$$= \left(\int_{-\pi}^0 \sin u du\right)\left(\int_0^{\frac{\pi}{2}} \cos w dw\right) = [-\cos u]_{-\pi}^0 \cdot [\sin w]_0^{\frac{\pi}{2}} = -2.$$

(4)　極座標変換より，

$$\iiint_U \sin\sqrt{x^2+y^2+z^2}dxdydz = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^{2\pi} r^2 \sin r \sin\theta dr$$
$$= 2\pi\left(\int_0^\pi \sin\theta d\theta\right)\left(\int_0^{2\pi} r^2 \sin r dr\right) = 2\pi[-\cos\theta]_0^\pi\left(\left[-r^2\cos r\right]_0^{2\pi} + 2\int_0^{2\pi} r\cos r dr\right)$$
$$= 4\left(-4\pi^2 + 2[r\sin r]_0^{2\pi} - 2\int_0^{2\pi} \sin r dr\right) = -16\pi^3.$$

(5)　円柱座標変換より，

$$\iiint_U \sin(x^2+y^2)dxdydz = \int_0^{\sqrt{\frac{\pi}{2}}} dr \int_0^{2\pi} d\theta \int_0^{r^2} \sin(r^2) r dz$$
$$= 2\pi\int_0^{\sqrt{\frac{\pi}{2}}} r^3 \sin(r^2) dr = 2\pi\int_0^{\sqrt{\frac{\pi}{2}}} \frac{-r^2}{2}\left(\cos(r^2)\right)' dr$$
$$= 2\pi\left\{\left[-\frac{r^2}{2}\cos(r^2)\right]_0^{\sqrt{\frac{\pi}{2}}} + \int_0^{\sqrt{\frac{\pi}{2}}} r\cos(r^2)dr\right\}$$
$$= 2\pi\left[\frac{\sin(r^2)}{2}\right]_0^{\sqrt{\frac{\pi}{2}}} = \pi.$$

(6)　$x = -\dfrac{u+3w}{2}$，$y = u+v+w$，$z = -\dfrac{u+w}{2}$ により，$W\colon 0 \leqq u \leqq 1, 0 \leqq v \leqq 1, 0 \leqq w \leqq 1$ は U に写り，$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} -\frac{1}{2} & 0 & -\frac{3}{2} \\ 1 & 1 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} = -\dfrac{1}{2}$ なので，

$$\iiint_U (x^2+y^2+z^2)dxdydz$$
$$= \frac{1}{2}\int_0^1 du \int_0^1 dv \int_0^1 \left(\frac{3}{2}u^2 + v^2 + \frac{7}{2}w^2 + 2uv + 2vw + 4wu\right) dw$$
$$= \frac{1}{2}\int_0^1 du \int_0^1 \left(\frac{3}{2}u^2 + v^2 + \frac{7}{6} + 2uv + v + 2u\right) dv$$
$$= \frac{1}{2}\int_0^1 \left(\frac{3}{2}u^2 + \frac{1}{3} + \frac{7}{6} + u + \frac{1}{2} + 2u\right) du = \frac{1}{2}\int_0^1 \left(\frac{3}{2}u^2 + 3u + 2\right) du = 2.$$

問 4

(1) U_n: $\dfrac{1}{n^2} \leqq x, y, z \leqq 1$ or $-1 \leqq x, y, z \leqq -\dfrac{1}{n}$ とすると，

$$\iiint_U \frac{dxdydz}{xyz} = \lim_{n\to\infty} \iiint_{U_n} \frac{dxdydz}{xyz} = \lim_{n\to\infty} \left(\int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{dx}{x} \right)^3 = \lim_{n\to\infty} (\log n)^3 = \infty$$

なので，発散する．

(2) U_n: $9 \leqq x^2 + y^2 + z^2 \leqq n^2$ とすると，極座標変換より，

$$\iiint_U \frac{dxdydz}{(x^2+y^2+z^2)^2} = \lim_{n\to\infty} \iiint_{U_n} \frac{dxdydz}{(x^2+y^2+z^2)^2}$$
$$= \lim_{n\to\infty} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_3^n \frac{\sin\theta}{r^2} dr = 2\pi \lim_{n\to\infty} [-\cos\theta]_0^{\pi} \cdot \left[-\frac{1}{r} \right]_3^n = \frac{4}{3}\pi.$$

(3) U_n: $|x| \leqq n$, $|y| \leqq n$, $|z| \leqq n$ とすると，対称性より，

$$\iiint_U x^2y^2z^2 e^{-|x|-|y|-|z|} dxdydz = \lim_{n\to\infty} \iiint_{U_n} x^2y^2z^2 e^{-|x|-|y|-|z|} dxdydz$$
$$= 8 \lim_{n\to\infty} \left(\int_0^n x^2 e^{-x} dx \right)^3 = 8 \lim_{n\to\infty} \left(\left[-x^2 e^{-x} \right]_0^n + 2 \int_0^n x e^{-x} dx \right)^3$$
$$= 64 \lim_{n\to\infty} \left(\left[-x e^{-x} \right]_0^n + \int_0^n e^{-x} dx \right)^3 = 64.$$

(4) U_n: $-n \leqq x + y + z \leqq -\dfrac{1}{n}$, $|y| \leqq n$, $|z| \leqq n$ とし，$x = u - v - w$，$y = v$，$z = w$ と変換して，

$$\iiint_U \frac{dxdydz}{(x+y+z)^3} = \lim_{n\to\infty} \iiint_{U_n} \frac{dxdydz}{(x+y+z)^3} = \lim_{n\to\infty} \int_{-n}^n dv \int_{-n}^n dw \int_{-n}^{-\frac{1}{n}} \frac{du}{u^3}$$
$$= \lim_{n\to\infty} 4n^2 \left[-\frac{1}{2u^2} \right]_{-n}^{-\frac{1}{n}} = \lim_{n\to\infty} 2n^2 \left(-n^2 + \frac{1}{n^2} \right) = -\infty$$

より，発散する．

(5) U_n: $\dfrac{1}{n} \leqq x \leqq 1$, $\dfrac{1}{n} \leqq y \leqq 1$, $\dfrac{1}{n} \leqq z \leqq 1$ とすると，

$$\iiint_U \frac{dxdydz}{(x+y+z)^{\frac{5}{2}}} = \lim_{n\to\infty} \iiint_{U_n} \frac{dxdydz}{(x+y+z)^{\frac{5}{2}}}$$
$$= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 dy \int_{\frac{1}{n}}^1 \frac{dz}{(x+y+z)^{\frac{5}{2}}}$$
$$= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 \left[-\frac{2}{3}(x+y+z)^{-\frac{3}{2}} \right]_{z=\frac{1}{n}}^{z=1} dy$$
$$= \frac{2}{3} \lim_{n\to\infty} \int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 \left\{ -(x+y+1)^{-\frac{3}{2}} + \left(x + y + \frac{1}{n} \right)^{-\frac{3}{2}} \right\} dy$$
$$= \frac{2}{3} \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \left[2(x+y+1)^{-\frac{1}{2}} - 2\left(x + y + \frac{1}{n} \right)^{-\frac{1}{2}} \right]_{y=\frac{1}{n}}^{y=1} dx$$
$$= \frac{4}{3} \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \left\{ (x+2)^{-\frac{1}{2}} - 2\left(x + 1 + \frac{1}{n} \right)^{-\frac{1}{2}} + \left(x + \frac{2}{n} \right)^{-\frac{1}{2}} \right\} dx$$

$$= \frac{4}{3} \lim_{n\to\infty} \left[2(x+2)^{\frac{1}{2}} - 4\left(x + 1 + \frac{1}{n} \right)^{\frac{1}{2}} + 2\left(x + \frac{2}{n} \right)^{\frac{1}{2}} \right]_{\frac{1}{n}}^{1}$$

$$= \frac{4}{3} \left(2\sqrt{3} - 4\sqrt{2} + 2 - 2\sqrt{2} + 4 \right) = \frac{8}{3}(\sqrt{3} - 3\sqrt{2} + 3).$$

(6) $U_n \colon \dfrac{1}{n^2} \leqq x^2 + y^2 + z^2 \leqq 1$ とし，極座標変換より，

$$\iiint_U \log(x^2+y^2+z^2)dxdydz = \lim_{n\to\infty} \iiint_{U_n} \log(x^2+y^2+z^2)dxdydz$$

$$= \lim_{n\to\infty} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{\frac{1}{n}}^{1} \left(\log r^2 \right) r^2 \sin\theta dr = 4\pi \lim_{n\to\infty} \left(\int_0^{\pi} \sin\theta d\theta \right) \left(\int_{\frac{1}{n}}^{1} r \log r dr \right)$$

$$= 4\pi \left[-\cos\theta \right]_0^{\pi} \lim_{n\to\infty} \left(\left[\frac{r^3}{3} \log r \right]_{\frac{1}{n}}^{1} - \frac{1}{3} \int_{\frac{1}{n}}^{1} r^2 dr \right) = \frac{8}{3}\pi \lim_{n\to\infty} \left[-\frac{r^3}{3} \right]_{\frac{1}{n}}^{1} = -\frac{8}{9}\pi.$$

5.5 重積分の応用

問 1

(1) 合成関数の偏微分より，$z_r = \cos\theta z_x + \sin\theta z_y$，$z_\theta = -r\sin\theta z_x + r\cos\theta z_y$ なので，$z_x = \cos\theta z_r - \dfrac{\sin\theta}{r} z_\theta$，$z_y = \sin\theta z_r + \dfrac{\cos\theta}{r} z_\theta$．したがって，$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+z_r^2+\dfrac{1}{r^2}z_\theta^2}$ であり，極座標変換より，

$$S = \iint_D \sqrt{1+z_x^2+z_y^2}dxdy = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} r\sqrt{1+z_r^2+\frac{1}{r^2}z_\theta^2}dr$$

が成り立つ．

(2) $x'(t) \neq 0$ なので，$x = x(t)$ の逆関数が存在する．したがって，適当な関数 $f(x)$ を用いて，$y = f(x)$ とかける．このとき，$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ なので，定理 9 で置換積分 $x = x(t)$ を行えば，

$$S = 2\pi \int_\alpha^\beta |y(t)|\sqrt{x'(t)^2+y'(t)^2}dt$$

が得られる．

問 2　求める曲面積を S とする．

(1)　$z = \sqrt{1-x^2-y^2}$，$D\colon x^2+y^2 \leqq 1$ とすると，$z_x = -\dfrac{x}{\sqrt{1-x^2-y^2}}$，$z_y = -\dfrac{y}{\sqrt{1-x^2-y^2}}$ なので，対称性と極座標変換より，

$$S = 2\iint_D \frac{dxdy}{\sqrt{1-x^2-y^2}} = 2\int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}}dr = 4\pi\left[-\sqrt{1-r^2}\right]_0^1 = 4\pi.$$

(2)　$D\colon x^2+y^2 \leqq 1$ とすると，$z_x = 2x$，$z_y = 2y$ なので，極座標変換より，

$$\begin{aligned} S &= \iint_D \sqrt{1+4x^2+4y^2}dxdy = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+4r^2}rdr \\ &= 2\pi\left[\frac{1}{12}\left(1+4r^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right). \end{aligned}$$

(3)　$z = \sqrt{4-x^2}$，$D\colon x^2+y^2 \leqq 4$ とすると，$z_x = -\dfrac{x}{\sqrt{4-x^2}}$，$z_y = 0$ なので，対称性より，

$$S = 2\iint_D \frac{2}{\sqrt{4-x^2}}dxdy = 16\int_0^2 dx \int_0^{\sqrt{4-x^2}} \frac{dy}{\sqrt{4-x^2}} = 16\int_0^2 dx = 32.$$

(4)　$y = \pm\sqrt{1-x^2}+2$ なので，$y' = \mp\dfrac{x}{\sqrt{1-x^2}}$．したがって，

$$\begin{aligned} S &= 2\pi\int_{-1}^1 \left(\sqrt{1-x^2}+2-\sqrt{1-x^2}+2\right)\frac{1}{\sqrt{1-x^2}}dx = 8\pi\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \\ &= 8\pi\left[\sin^{-1}x\right]_{-1}^1 = 8\pi^2. \end{aligned}$$

(5)　$y' = -\sin x$ より，

$$S = 2\pi\int_0^\pi |\cos x|\sqrt{1+\sin^2 x}dx = 4\pi\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin^2 x}dx.$$

$t = \sin x$ と変換して，

$$S = 4\pi\int_0^1 \sqrt{t^2+1}dt = 4\pi\left[\frac{1}{2}\left\{t\sqrt{t^2+1}+\log\left(t+\sqrt{t^2+1}\right)\right\}\right]_0^1 = 2\pi\left\{\sqrt{2}+\log\left(1+\sqrt{2}\right)\right\}.$$

(6) $y=\pm\sqrt{1-\dfrac{x^2}{4}}$ なので，$y'=\mp\dfrac{x}{2\sqrt{4-x^2}}$ より，

$$
\begin{aligned}
S&=\frac{\pi}{2}\int_{-2}^{2}\sqrt{16-3x^2}dx=\sqrt{3}\pi\int_0^2\sqrt{\frac{16}{3}-x^2}dx\\
&=\sqrt{3}\pi\left[\frac{1}{2}\left(x\sqrt{\frac{16}{3}-x^2}+\frac{16}{3}\sin^{-1}\frac{\sqrt{3}}{4}x\right)\right]_0^2=2\pi\left(1+\frac{4}{3\sqrt{3}}\pi\right).
\end{aligned}
$$

問 3 物体 U の全質量を M，重心を $G(x_0,y_0,z_0)$，z 軸に関する慣性モーメントを I_z とする．

(1) $M=\iiint_U\rho dxdydz=48\rho$．対称性より，$G(0,0,0)$．対称性より，

$$
\begin{aligned}
I_z&=\iiint_U(x^2+y^2)\rho dxdydz=8\rho\int_0^1dx\int_0^2dy\int_0^3(x^2+y^2)dz\\
&=24\rho\int_0^1dx\int_0^2(x^2+y^2)dy=24\rho\int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=2}dx\\
&=24\rho\int_0^1\left(2x^2+\frac{8}{3}\right)dx=24\rho\left[\frac{2}{3}x^3+\frac{8}{3}\right]_0^1=80\rho.
\end{aligned}
$$

(2)
$$
\begin{aligned}
M&=\iiint_U\rho dxdydz=\rho\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}dz\\
&=\rho\int_0^1dx\int_0^{1-x}(1-x-y)dy=\rho\int_0^1\left[(1-x)y-\frac{y^2}{2}\right]_{y=0}^{y=1-x}dx\\
&=\frac{1}{2}\rho\int_0^1(1-x)^2dx=\frac{1}{2}\rho\left[-\frac{(1-x)^3}{3}\right]_0^1=\frac{\rho}{6}.
\end{aligned}
$$

$$
\begin{aligned}
x_0&=\frac{1}{M}\iiint_U x\rho dxdydz=6\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}xdz\\
&=6\int_0^1dx\int_0^{1-x}x(1-x-y)dy=6\int_0^1\left[x(1-x)y-\frac{x}{2}y^2\right]_{y=0}^{y=1-x}dx\\
&=3\int_0^1(x-2x^2+x^3)dx=3\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1=\frac{1}{4}.
\end{aligned}
$$

同様にして，$y_0=z_0=\dfrac{1}{4}$．よって，$G\left(\dfrac{1}{4},\dfrac{1}{4},\dfrac{1}{4}\right)$.

$$
\begin{aligned}
I_z&=\iiint_U(x^2+y^2)\rho dxdydz=\rho\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}(x^2+y^2)dz\\
&=\rho\int_0^1dx\int_0^{1-x}(x^2-x^3-x^2y+(1-x)y^2-y^3)dy\\
&=\rho\int_0^1\left[(x^2-x^3)y-\frac{x^2}{2}y^2+\frac{1-x}{3}y^3-\frac{y^4}{4}\right]_{y=0}^{y=1-x}\\
&=\frac{\rho}{12}\int_0^1(7x^4-16x^3+12x^2-4x+1)dx=\frac{\rho}{12}\left[\frac{7}{5}x^5-4x^4+4x^3-2x^2+x\right]_0^1=\frac{\rho}{30}.
\end{aligned}
$$

(3) $M=\iiint_U\rho dxdydz=\rho\int_1^2dx\int_0^{2-x}dy\int_1^{3-x-y}dz=\rho\int_1^2dx\int_0^{2-x}(2-x-y)dy$

$$= \rho \int_1^2 \left[(2-x)y - \frac{y^2}{2} \right]_{y=0}^{y=2-x} dx = \frac{\rho}{2} \int_1^2 (2-x)^2 dx = \frac{\rho}{2} \left[-\frac{(2-x)^3}{3} \right]_1^2 = \frac{\rho}{6}.$$

$$\begin{aligned} x_0 &= \frac{1}{M} \iiint_U x\rho dxdydz = 6 \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} xdz \\ &= 6 \int_1^2 dx \int_0^{2-x} (2x - x^2 - yx) dy = 6 \int_1^2 \left[(2x-x^2)y - \frac{x}{2} y^2 \right]_{y=0}^{y=2-x} dx \\ &= 3 \int_1^2 (4x - 4x^2 + x^3) dx = 3 \left[2x^2 - \frac{4}{3} x^3 + \frac{x^4}{4} \right]_1^2 = \frac{5}{4}. \end{aligned}$$

同様にして，$z_0 = \dfrac{5}{4}$.

$$\begin{aligned} y_0 &= \frac{1}{M} \iiint_U y\rho dxdydz = 6 \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} ydz \\ &= 6 \int_1^2 dx \int_1^{2-x} \left\{ (2-x)y - y^2 \right\} dy = 6 \int_1^2 \left[\frac{2-x}{2} y^2 - \frac{y^3}{3} \right]_{y=0}^{y=2-x} dx \\ &= \int_1^2 (2-x)^3 dx = \left[-\frac{(2-x)^4}{4} \right]_1^2 = \frac{1}{4}. \end{aligned}$$

よって，$G\left(\dfrac{5}{4}, \dfrac{1}{4}, \dfrac{5}{4}\right)$.

$$\begin{aligned} I_z &= \iiint_U (x^2 + y^2) \rho dxdydz = \rho \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} (x^2+y^2) dz \\ &= \rho \int_1^2 dx \int_0^{2-x} \left\{ 2x^2 - x^3 - x^2 y + (2-x)y^2 - y^3 \right\} dy \\ &= \rho \int_1^2 \left[(2x^2 - x^3)y - \frac{x^2}{2} y^2 + \frac{2-x}{3} y^3 - \frac{y^4}{4} \right]_{y=0}^{y=2-x} dx \\ &= \frac{\rho}{12} \int_1^2 (7x^4 - 32x^3 + 48x^2 - 32x + 16) dx \\ &= \frac{\rho}{12} \left[\frac{7}{5} x^5 - 8x^4 + 16x^3 - 16x^2 + 16x \right]_1^2 = \frac{17}{60} \rho. \end{aligned}$$

(4)　$D\colon x^2 + y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0$ とする.

$$M = \iiint_U \rho dxdydz = \rho \int_0^1 dz \iint_D dxdy = \frac{\pi}{4} \rho.$$

$$\begin{aligned} x_0 &= \frac{1}{M} \iiint_U x\rho dxdydz = \frac{4}{\pi} \int_0^1 dz \iint_D xdxdy = \frac{4}{\pi} \int_0^1 dy \int_0^{\sqrt{1-y^2}} xdx \\ &= \frac{4}{\pi} \int_0^1 \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \frac{2}{\pi} \int_0^1 (1-y^2) dy = \frac{2}{\pi} \left[y - \frac{y^3}{3} \right]_0^1 = \frac{4}{3\pi}. \end{aligned}$$

同様にして，$y_0 = \dfrac{4}{3\pi}$. $z_0 = \dfrac{1}{M} \displaystyle\iiint_U z\rho dxdydz = \dfrac{4}{\pi} \int_0^1 dz \iint_D zdxdy = \int_0^1 zdz = \dfrac{1}{2}$. よって，$G\left(\dfrac{4}{3\pi}, \dfrac{4}{3\pi}, \dfrac{1}{2}\right)$. 極座標変換より，

$$I_z = \iiint_U (x^2+y^2) \rho dxdydz = \rho \int_0^1 dz \iint_D (x^2+y^2) dxdy = \rho \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 dr = \frac{\pi}{8} \rho.$$

(5)　$x=u-1$，$v=v+1$，$z=w$ により $W\colon u^2+v^2 \leqq 1,\ 0 \leqq w \leqq 1$ は U に写る．$E\colon u^2+v^2 \leqq 1$ とおく．

$$M=\iiint_U \rho dxdydz=\rho\iiint_W dudvdw=\rho\int_0^1 dw\iint_E dudv=\pi\rho.$$

$x_0=\dfrac{1}{M}\displaystyle\iiint_W (u-1)\rho dudvdw=-1$．同様にして，$y_0=1$．

$$z_0=\frac{1}{M}\iiint_U z\rho dzdydz=\frac{1}{\pi}\int_0^1 dw\iint_E wdudv=\frac{1}{2}.$$

よって，$G\left(-1,1,\dfrac{1}{2}\right)$．極座標変換より，

$$\begin{aligned}I_z&=\iiint_U (x^2+y^2)\rho dxdydz=\rho\int_0^1 dw\iint_D \left\{(u-1)^2+(v+1)^2\right\}dudv\\&=\rho\int_0^{2\pi}d\theta\int_0^1\left\{r^3-2r^2(\cos\theta+\sin\theta)+2r\right\}dr\\&=\rho\int_0^{2\pi}\left[\frac{r^4}{4}-\frac{r^3}{3}(\cos\theta+\sin\theta)+r^2\right]_{r=0}^{r=1}d\theta\\&=\rho\int_0^{2\pi}\left\{\frac{5}{4}-\frac{1}{3}(\cos\theta+\sin\theta)\right\}d\theta=\frac{5}{2}\pi\rho.\end{aligned}$$

(6)　極座標変換より，

$$\begin{aligned}M&=\iiint_U \rho dxdydz=\rho\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^2\sin\theta dr\\&=\frac{\pi}{2}\rho\left(\int_0^{\frac{\pi}{2}}\sin\theta d\theta\right)\left(\int_0^1 r^2dr\right)=\frac{\pi}{2}\rho\left[-\cos\theta\right]_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^1=\frac{\pi}{6}\rho.\end{aligned}$$

$$\begin{aligned}x_0&=\frac{1}{M}\iiint_U x\rho dxdydz=\frac{6}{\pi}\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^3\sin^2\theta\cos\varphi dr\\&=\frac{6}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos\varphi d\varphi\right)\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}d\theta\right)\left(\int_0^1 r^3dr\right)\\&=\frac{3}{\pi}\left[\sin\varphi\right]_0^{\frac{\pi}{2}}\cdot\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^4}{4}\right]_0^1=\frac{3}{8}.\end{aligned}$$

同様にして，$y_0=z_0=\dfrac{3}{8}$．よって，$G\left(\dfrac{3}{8},\dfrac{3}{8},\dfrac{3}{8}\right)$．

$$\begin{aligned}I_z&=\iiint_U (x^2+y^2)\rho dxdydz=\rho\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^4\sin^3\theta dr\\&=\frac{\pi}{2}\rho\left(\int_0^{\frac{\pi}{2}}\frac{3\sin\theta-\sin 3\theta}{4}d\theta\right)\left(\int_0^1 r^4dr\right)=\frac{\pi}{8}\rho\left[-3\cos\theta+\frac{\cos 3\theta}{3}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^5}{5}\right]_0^1=\frac{\pi}{15}\rho.\end{aligned}$$

演習問題 5

1. 求める面積を S とおく.

(1)　$(x+y+1)^2+y^2=1$ が囲む面積を求めればよい．$u=x+y+1$，$v=y$ により，$E\colon u^2+v^2\leqq 1, u\geqq 0, v\geqq 0$ は $D\colon x^2+2y^2+2xy+2x+2y\leqq 0, x+y+1\geqq 0, y\geqq 0$ に写り，$\dfrac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=1$．対称性より，

$$S=4\iint_D dxdy=4\iint_E dudv=4\int_0^1 dr\int_0^{\frac{\pi}{2}} rd\theta=\pi.$$

(2)　$3x^2-8xy-3y^2=1$ は $(2x-y)^2-(x+2y)^2=1$ とかける．$u=2x-y$，$v=x+2y$ により，$E\colon u^2-v^2\geqq 1, u\leqq\sqrt{2}$ は $D\colon 3x^2-8xy-3y^2\geqq 1, y\geqq 2x-\sqrt{2}$ に写り，$\dfrac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}=5$. 対称性より，

$$\begin{aligned}S&=\iint_D dxdy=\iint_E \frac{1}{5}dudv=\frac{2}{5}\int_1^{\sqrt{2}}du\int_0^{\sqrt{u^2-1}}dv=\frac{2}{5}\int_1^{\sqrt{2}}\sqrt{u^2-1}du\\&=\frac{1}{5}\left[u\sqrt{u^2-1}-\log\left(u+\sqrt{u^2-1}\right)\right]_1^{\sqrt{2}}=\frac{\sqrt{2}-\log\left(\sqrt{2}+1\right)}{5}.\end{aligned}$$

(3)　$x=r\cos\theta$，$y=r\sin\theta$ により，$E\colon 0\leqq r\leqq\sqrt{\dfrac{\sin 2\theta}{2}}, 0\leqq\theta\leqq\dfrac{\pi}{4}$ は $D\colon (x^2+y^2)^2\leqq xy, x\geqq y\geqq 0$ に写るので，対称性より，

$$S=4\iint_D dxdy=4\int_0^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{\frac{\sin 2\theta}{2}}}rdr=\int_0^{\frac{\pi}{4}}\sin 2\theta d\theta=\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{1}{2}.$$

(4)　$D\colon \sqrt{x}+\sqrt{y}\leqq 1, \sqrt{x-1}+\sqrt{y}\leqq 1, x\geqq 0, y\geqq 0$ とおくと，対称性より，

$$\begin{aligned}S&=4\iint_D dxdy=4\int_0^{\frac{1}{2}}\left(1-\sqrt{1-x}\right)^2dx\\&=4\int_0^{\frac{1}{2}}\left(1-2\sqrt{1-x}+1-x\right)dx=4\int_0^{\frac{1}{2}}\left(2-x-2\sqrt{1-x}\right)dx\\&=4\left[2x-\frac{x^2}{2}+\frac{4}{3}(1-x)^{\frac{3}{2}}\right]_0^{\frac{1}{2}}=4\left(1-\frac{1}{8}+\frac{2}{3\sqrt{2}}-\frac{4}{3}\right)\\&=\frac{4}{3}\sqrt{2}-\frac{11}{6}.\end{aligned}$$

(5)　$x=r\cos\theta$，$y=r\sin\theta$ により，$E\colon 0\leqq r\leqq\dfrac{3\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}, 0\leqq\theta\leqq\dfrac{\pi}{4}$ は $D\colon x^3+y^3\leqq 3xy, x\geqq y\geqq 0$ に写るので，対称性より，

$$\begin{aligned}S&=2\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{3\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}}rdr=9\int_0^{\frac{\pi}{4}}\frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}d\theta\\&=9\int_0^{\frac{\pi}{4}}\frac{\sin^2\theta\cos^2\theta}{\sin^6\theta+\cos^6\theta+2\sin^3\theta\cos^3\theta}d\theta.\end{aligned}$$

ここで，$t=\tan\theta$ とおくと，

$$S=9\int_0^1\frac{t^2}{1+t^6+2t^3}dt=9\int_0^1\frac{t^2}{(t^3+1)^2}dt=3\left[-\frac{1}{t^3+1}\right]_0^1=\frac{3}{2}.$$

2.

(1)　$x=\dfrac{r\cos\theta-1}{2}$，$y=\dfrac{r\sin\theta+2}{3}$ により，$E\colon 0\leqq r\leqq\sqrt{5},\ 0\leqq\theta\leqq 2\pi$ は $D\colon 4\left(x+\dfrac{1}{2}\right)^2+9\left(y-\dfrac{2}{3}\right)^2\leqq 5$ に写り，$\dfrac{\partial(x,y)}{\partial(r,\theta)}=\det\begin{pmatrix}\frac{\cos\theta}{2} & -\frac{r}{2}\sin\theta\\ \frac{\sin\theta}{3} & \frac{r}{3}\cos\theta\end{pmatrix}=\dfrac{r}{6}$ なので，

$$\begin{aligned}\iint_D xydxdy&=\frac{1}{36}\iint_E r(r\cos\theta-1)(r\sin\theta+2)drd\theta\\&=\frac{1}{36}\int_0^{\sqrt{5}}dr\int_0^{2\pi}r\left\{r^2\frac{\sin 2\theta}{2}+r(2\cos\theta-\sin\theta)-2\right\}d\theta\\&=\frac{1}{36}\int_0^{\sqrt{5}}r\left[-r^2\frac{\cos 2\theta}{2}+r(2\sin\theta+\cos\theta)-2\theta\right]_{\theta=0}^{\theta=2\pi}dr\\&=-\frac{\pi}{9}\int_0^{\sqrt{5}}rdr=-\frac{5}{18}\pi.\end{aligned}$$

(2)　$D_n\colon 1\leqq x\leqq n,\ 0\leqq y\leqq\log x$ とすると，

$$\iint_D\frac{y}{x^2}dxdy=\lim_{n\to\infty}\int_1^n dx\int_0^{\log x}\frac{y}{x^2}dy=\lim_{n\to\infty}\int_1^n\frac{1}{x^2}\left[\frac{y^2}{2}\right]_{y=0}^{y=\log x}dx=\frac{1}{2}\lim_{n\to\infty}\int_1^n\frac{\log^2 x}{x^2}dx.$$

ここで，$x=e^t$ と変換して，

$$\begin{aligned}\iint_D\frac{y}{x^2}dxdy&=\frac{1}{2}\lim_{n\to\infty}\int_0^{\log n}t^2e^{-t}dt=\frac{1}{2}\lim_{n\to\infty}\left(\left[-t^2e^{-t}\right]_0^{\log n}+\int_0^{\log n}2te^{-t}dt\right)\\&=\lim_{n\to\infty}\int_0^{\log n}te^{-t}dt=\lim_{n\to\infty}\left(\left[-te^{-t}\right]_0^{\log n}+\int_0^{\log n}e^{-t}dt\right)\\&=\lim_{n\to\infty}\left[-e^{-t}\right]_0^{\log n}=1.\end{aligned}$$

(3)　$D_n\colon \dfrac{1}{n}\leqq|x|\leqq 1,\ |x|+|y|\leqq 1$ とすると，対称性より，

$$\begin{aligned}&\iint_D\frac{dxdy}{\sqrt{x^2+y^2}}dxdy=\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{\sqrt{x^2+y^2}}=4\lim_{n\to\infty}\int_{\frac{1}{n}}^1dx\int_0^{1-x}\frac{dy}{\sqrt{x^2+y^2}}\\&=4\lim_{n\to\infty}\int_{\frac{1}{n}}^1\left[\log\left(y+\sqrt{x^2+y^2}\right)\right]_{y=0}^{y=1-x}dx\\&=4\lim_{n\to\infty}\int_{\frac{1}{n}}^1\left\{\log\left(1-x+\sqrt{2x^2-2x+1}\right)-\log x\right\}dx\\&=4\lim_{n\to\infty}\left\{\left[x\left\{\log\left(1-x+\sqrt{2x^2-2x+1}\right)-\log x\right\}\right]_{\frac{1}{n}}^1+\int_{\frac{1}{n}}^1\frac{dx}{\sqrt{2x^2-2x+1}}\right\}\\&=4\left[-\frac{1}{\sqrt{2}}\log\left|1-2x+\sqrt{4x^2-4x+2}\right|\right]_0^1=4\sqrt{2}\log\left(1+\sqrt{2}\right).\end{aligned}$$

(4)　$D_n\colon 0\leqq x\leqq n,\ 0\leqq y\leqq n$ とすると，対称性より，

$$\begin{aligned}&\iint_D\frac{dxdy}{e^x+e^y}=\lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{e^x+e^y}=2\lim_{n\to\infty}\int_0^n dx\int_0^x\frac{dy}{e^x+e^y}\\&=2\lim_{n\to\infty}\int_0^n\left[\frac{y-\log(e^x+e^y)}{e^x}\right]_{y=0}^{y=x}dx=2\lim_{n\to\infty}\int_0^n e^{-x}\left\{-\log 2+\log(e^x+1)\right\}dx\end{aligned}$$

$$= 2\lim_{n\to\infty}\left\{\left[-e^{-x}\left\{-\log 2+\log(e^x+1)\right\}\right]_0^n+\int_0^n\frac{dx}{e^x+1}\right\}$$
$$= 2\lim_{n\to\infty}\left[x-\log(e^x+1)\right]_0^n = 2\log 2.$$

(5)　$D_n\colon 0 \leqq y \leqq \pi-\dfrac{2}{n},\ y+\dfrac{1}{n}\leqq x\leqq \pi-\dfrac{1}{n}$ とすると，

$$\iint_D \frac{\sin y}{\sqrt{(\pi-x)(x-y)}}dxdy = \lim_{n\to\infty}\iint_{D_n}\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}dxdy$$
$$= \lim_{n\to\infty}\int_0^{\pi-\frac{2}{n}}dy\int_{y+\frac{1}{n}}^{\pi-\frac{1}{n}}\frac{\sin y}{\sqrt{\left(\frac{\pi-y}{2}\right)^2-\left(x-\frac{y+\pi}{2}\right)^2}}dx$$
$$= \lim_{n\to\infty}\int_0^{\pi-\frac{2}{n}}\sin y\left[\sin^{-1}\frac{2x-\pi-y}{\pi-y}\right]_{x=y+\frac{1}{n}}^{x=\pi-\frac{1}{n}}dy$$
$$= 2\lim_{n\to\infty}\int_0^{\pi-\frac{2}{n}}(\sin y)\sin^{-1}\frac{\pi-\frac{2}{n}-y}{\pi-y}dy$$
$$= 2\lim_{n\to\infty}\int_{\frac{2}{n}}^{\pi}(\sin z)\sin^{-1}\frac{z-\frac{2}{n}}{z}dz \quad (z=\pi-y).$$

ここで，$z\geqq\dfrac{2}{n}$ のとき，

$$\left|\sin^{-1}\frac{z-\frac{2}{n}}{z}-\frac{\pi}{2}\right| = \left|\sin^{-1}\left(1-\frac{2}{nz}\right)-\sin^{-1}1\right| = \left|\frac{2}{nz}\int_0^1\frac{d\theta}{\sqrt{1-(1-\frac{2\theta}{nz})^2}}\right|$$
$$= \left|\frac{2}{nz}\int_0^1\frac{d\theta}{\sqrt{(2-\frac{2\theta}{nz})\frac{2\theta}{nz}}}\right| \leqq \frac{1}{\sqrt{nz-1}}\int_0^1\frac{d\theta}{\sqrt{\theta}} = \frac{2}{\sqrt{nz-1}}.$$

故に，

$$\int_{\frac{2}{n}}^{\pi}\frac{2}{\sqrt{nz-1}}dz = \left[4\frac{\sqrt{nz-1}}{n}\right]_{\frac{2}{n}}^{\pi} = 4\frac{\sqrt{\pi n-1}-1}{n}\to 0 \quad (n\to\infty).$$

従って，

$$\iint_D\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}dxdy = 2\int_0^{\pi}\sin z\cdot\frac{\pi}{2}dz = \pi[-\cos z]_0^{\pi} = 2\pi.$$

3. 求める体積を V とおく．

(1)　$(x+y)^2+y^2+(y+2z)^2=1$ であり，$u=x+y,\ v=y,\ w=y+2z$ とおけば $W\colon u^2+v^2+z^2=1$ は U に写る．また，$\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\det\begin{pmatrix}1&1&0\\0&1&0\\0&1&2\end{pmatrix}=2$. 故に，

$$V=\iiint_U dxdydz=\frac{1}{2}\iiint_W dudvdw=\frac{1}{2}\int_0^1 dr\int_0^{\pi}d\theta\int_0^{2\pi}r^2\sin\theta d\varphi=\frac{1}{2}\frac{1}{3}\cdot 2\cdot 2\pi=\frac{2}{3}\pi.$$

(2)　$\left(x+\dfrac{y}{2}\right)^2+\dfrac{3}{4}y^2\leqq 1,\ \dfrac{3}{4}y^2+\left(-\dfrac{y}{2}+z\right)^2\leqq 1$ であり，$u=x+\dfrac{y}{2},\ v=\dfrac{\sqrt{3}}{2}y,\ w=-\dfrac{y}{2}+z$ とおけば，$u^2+v^2\leqq 1,\ v^2+w^2\leqq 1$. また，$\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\det\begin{pmatrix}1&\frac{1}{2}&0\\0&\frac{\sqrt{3}}{2}&0\\0&-\frac{1}{2}&1\end{pmatrix}=\dfrac{\sqrt{3}}{2}$. 故に，

$$V=\iiint_U dxdydz=\frac{2}{\sqrt{3}}\iiint_W dudvdw=\frac{16}{\sqrt{3}}\int_0^1 dv\int_0^{\sqrt{1-v^2}}du\int_0^{\sqrt{1-v^2}}dw$$

$$= \frac{16}{\sqrt{3}}\int_0^1 (1-v^2)dv = \frac{16}{\sqrt{3}}\left[v - \frac{v^3}{3}\right]_0^1 = \frac{32}{3\sqrt{3}}.$$

(3)　対称性より，

$$\begin{aligned} V &= 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} dy \int_0^{(1-\sqrt{y}-\sqrt{z})^2} dx = 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} (1-\sqrt{y}-\sqrt{z})^2 dy \\ &= 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} \left\{y - 2(1-\sqrt{z})\sqrt{y} + (1-\sqrt{z})^2\right\} dy \\ &= 8\int_0^1 \left[\frac{y^2}{2} - \frac{4}{3}(1-\sqrt{z})y^{\frac{3}{2}} + (1-\sqrt{z})^2 y\right]_{y=0}^{y=(1-\sqrt{z})^2} dz = \frac{4}{3}\int_0^1 (1-\sqrt{z})^4 dz. \end{aligned}$$

ここで，$z = t^2$ とすると $\dfrac{dz}{dt} = 2t$ なので，

$$\begin{aligned} V &= \frac{8}{3}\int_0^1 t(1-t)^4 dt = \frac{8}{3}\int_0^1 \left\{(1-t)^4 - (1-t)^5\right\} dt \\ &= \frac{8}{3}\left[-\frac{1}{5}(1-t)^5 + \frac{1}{6}(1-t)^6\right]_0^1 = \frac{4}{45}. \end{aligned}$$

(4)　$A(z)\colon x^2 + y^2 \leqq \left(1 - z^{\frac{2}{3}}\right)^3$ とおき，$x = r\cos\theta$，$y = r\sin\theta$ とすると，

$$\begin{aligned} V &= \int_{-1}^1 dz \iint_{A(z)} dxdy = 2\left(\int_0^1 dz \int_0^{(1-z^{\frac{2}{3}})^{\frac{3}{2}}} rdr\right)\left(\int_0^{2\pi} d\theta\right) = 2\pi\int_0^1 \left(1 - z^{\frac{2}{3}}\right)^3 dz \\ &= 2\pi\int_0^1 \left(1 - 3z^{\frac{2}{3}} + 3z^{\frac{4}{3}} - z^2\right) dz = 2\pi\left[z - \frac{9}{5}z^{\frac{5}{3}} + \frac{9}{7}z^{\frac{7}{3}} - \frac{z^3}{3}\right]_0^1 = \frac{32}{105}\pi. \end{aligned}$$

(5)　$x = r\cos\theta$，$y = r\sin\theta$ とすると，$0 \leqq \theta \leqq \pi$ で，$r = \cos\theta + 1$ なので，$x = \cos\theta(\cos\theta + 1)$，$y = \sin\theta(\cos\theta + 1)$．$x' = \dfrac{dx}{d\theta} = -\sin\theta(1 + 2\cos\theta)$ より，

$$\begin{aligned} V &= \pi\int_\pi^0 y^2 x' d\theta = \pi\int_0^\pi \sin^3\theta(\cos\theta + 1)^2(2\cos\theta + 1)d\theta \\ &= \pi\int_{-1}^1 (1-t^2)(t+1)^2(2t+1)dt \quad (t = \cos\theta) \\ &= \pi\int_{-1}^1 (-2t^5 - 5t^4 - 2t^3 + 4t^2 + 4t + 1)dt \\ &= 2\pi\int_0^1 (-5t^4 + 4t^2 + 1)dt = 2\pi\left[-t^5 + \frac{4}{3}t^3 + t\right]_0^1 = \frac{8}{3}\pi. \end{aligned}$$

4.

(1)　対称性より，

$$\begin{aligned} &\iiint_U \frac{dxdydz}{(|x|+|y|+|z|+1)^3} = 8\int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} \frac{dx}{(x+y+z+1)^3} \\ &= 8\int_0^1 dz \int_0^{1-z} \left[-\frac{1}{2(x+y+z+1)^2}\right]_{z=0}^{z=1-x-y} dy \\ &= 4\int_0^1 dz \int_0^{1-z} \left\{\frac{1}{(x+y+1)^2} - \frac{1}{4}\right\} dy \end{aligned}$$

$$= 4\int_0^1 \left[-\frac{1}{x+y+1} - \frac{y}{4}\right]_{y=0}^{y=1-z} dz = 4\int_0^1 \left(\frac{1}{z+1} + \frac{z}{4} - \frac{3}{4}\right) dz$$
$$= 4\left[\log(z+1) + \frac{z^2}{8} - \frac{3}{4}z\right]_0^1 = 4\log 2 - \frac{5}{2}.$$

(2)　$U_n\colon x^2+y^2+z^2 \leqq \sqrt{1-\frac{1}{n}}$ とすると，対称性と極座標変換より，

$$\iiint_U \frac{|z|}{\sqrt{1-x^2-y^2-z^2}}dxdydz = \lim_{n\to\infty}\iiint_{U_n} \frac{|z|}{\sqrt{1-x^2-y^2-z^2}}dxdydz$$
$$= 8\lim_{n\to\infty}\int_0^{1-\frac{1}{n}} dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \frac{r^3\sin\theta\cos\theta}{\sqrt{1-r^2}}d\varphi$$
$$= 2\pi\lim_{n\to\infty}\left(\int_0^{1-\frac{1}{n}} \frac{r^3}{\sqrt{1-r^2}}dr\right)\left(\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta\right)$$
$$= 2\pi\lim_{n\to\infty}\left(\left[-r^2\sqrt{1-r^2}\right]_0^{1-\frac{1}{n}} + 2\int_0^{1-\frac{1}{n}} r\sqrt{1-r^2}dr\right)\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$
$$= 4\pi\left[-\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{8}{3}\pi.$$

(3)　$U_n\colon \frac{1}{n} \leqq |x| \leqq 1,\ \frac{1}{n} \leqq |y| \leqq 1,\ \frac{1}{n} \leqq |z| \leqq 1$ とすると，対称性より，

$$\iiint_U \log|xyz|dxdydz = \lim_{n\to\infty}\iiint_{U_n} \log|xyz|dxdydz$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 dy \int_{\frac{1}{n}}^1 \log(xyz)dz$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 \left[\log(xy)z + z\left(\log z - 1\right)\right]_{z=\frac{1}{n}}^{z=1} dy$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx \int_{\frac{1}{n}}^1 \left\{\log(xy) - 1\right\} dy$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left[\left(\log x - 1\right)y + y\left(\log y - 1\right)\right]_{y=\frac{1}{n}}^{y=1} dx$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left(\log x - 2\right)dx = 8\lim_{n\to\infty}\left[x\left(\log x - 1\right) - 2x\right]_{\frac{1}{n}}^1 = -24.$$

(4)　対称性より，

$$\iiint_U x^2y^2z^2dxdydz = 8\int_0^1 dy \int_0^y dz \int_0^{y-z} x^2y^2z^2dx$$
$$= 8\int_0^1 dy \int_0^y y^2z^2\left[\frac{x^3}{3}\right]_{x=0}^{x=y-z} dy = \frac{8}{3}\int_0^1 dy \int_0^y (y-z)^3y^2z^2dz.$$

ここで，$z = y\tilde{z}$ とすると，

$$\iiint_U x^2y^2z^2dxdydz = \frac{8}{3}\int_0^1 dy \int_0^1 \tilde{z}^2(1-\tilde{z})^3y^8d\tilde{z}$$
$$= \frac{8}{3}\left(\int_0^1 y^8dy\right)\left(\int_0^1 \left(\tilde{z}^2 - 3\tilde{z}^3 + 3\tilde{z}^4 - \tilde{z}^5\right)d\tilde{z}\right)$$

$$= \frac{8}{3}\left[\frac{y^9}{9}\right]_0^1 \cdot \left[\frac{z^3}{3} - \frac{3}{4}z^4 + \frac{3}{5}z^5 - \frac{z^6}{6}\right]_0^1 = \frac{2}{405}.$$

(5) $U_n\colon \frac{1}{n^2} \leqq x^2+y^2+\frac{z^2}{9} \leqq 1$ とし，$x = r\sin\theta\cos\varphi$，$y = r\sin\theta\sin\varphi$，$z = 3r\cos\theta$ とすると，対称性より，

$$\begin{aligned}
\iiint_U \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} &= \lim_{n\to\infty}\iiint_{U_n} \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} \\
&= 24\lim_{n\to\infty}\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} \frac{r\sin\theta}{\sqrt{1+8\cos^2\theta}}d\varphi \\
&= 12\pi\left(\int_0^1 rdr\right)\left(\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta\right) \\
&= 6\pi\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta.
\end{aligned}$$

ここで，$t = \cos\theta$ と置換すると，

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta = \int_0^1 \frac{dt}{\sqrt{1+8t^2}} = \left[\frac{\sinh^{-1}(2\sqrt{2}t)}{2\sqrt{2}}\right]_0^1 = \frac{\sinh^{-1}(2\sqrt{2})}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$$

なので，$\iiint_U \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = 3\sqrt{2}\pi\log\left(1+\sqrt{2}\right)$.

5. 求める曲面積を S とする．

(1) 対称性より，$z \geqq 0$ としてよい．$(x^2+y^2)^{\frac{1}{3}} + z^{\frac{2}{3}} = 1$ の両辺を x，y で偏微分して，

$$z_x = -\frac{xz^{\frac{1}{3}}}{(x^2+y^2)^{\frac{2}{3}}},\quad z_y = -\frac{yz^{\frac{1}{3}}}{(x^2+y^2)^{\frac{2}{3}}}$$

なので，$\sqrt{1+z_x^2+z_y^2} = (x^2+y^2)^{-\frac{1}{6}}$．$D\colon x^2+y^2 \leqq 1$ として，極座標変換より，

$$S = 2\iint_D (x^2+y^2)^{-\frac{1}{6}}dxdy = 2\int_0^1 dr\int_0^{2\pi} r^{\frac{2}{3}}d\theta = 4\pi\left[\frac{3}{5}r^{\frac{5}{3}}\right]_0^1 = \frac{12}{5}\pi.$$

(2) $x = r\cos\theta$，$y = r\sin\theta$ とすると，$(x^2+y^2)^2 = x^2-y^2$ の第 1 象限にある部分は $r = \sqrt{\cos 2\theta}$ $\left(0 \leqq \theta \leqq \frac{\pi}{4}\right)$ とかける．このとき，

$$\begin{aligned}
&\frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}},\quad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = -\frac{1}{\sqrt{\cos 2\theta}}\sin 3\theta, \\
&\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = \frac{1}{\sqrt{\cos 2\theta}}\cos 3\theta,\quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)^2 = \frac{1}{\sin^2 3\theta}
\end{aligned}$$

なので，

$$S = 2\pi\int_0^{\frac{\pi}{4}} \sin\theta d\theta = 2\pi\left[-\cos\theta\right]_0^{\frac{\pi}{4}} = \left(2-\sqrt{2}\right)\pi.$$

(3) $x = r\cos\theta$，$y = r\sin\theta$ とすると，$r = 1+\cos\theta$ $(0 \leqq \theta \leqq \pi)$ とかけるので，$x = \cos\theta + \cos^2\theta$，$y = \sin\theta + \sin\theta\cos\theta$．従って，$\frac{dx}{d\theta} = -\sin\theta - \sin 2\theta$，$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$ であり，

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)^2 = \frac{4\cos^2\frac{\theta}{2}}{(\sin\theta+\sin 2\theta)^2}$$

なので，

$$S = 2\pi \int_0^\pi \sin\theta(1+\cos\theta)2\cos\frac{\theta}{2}d\theta = 8\pi\int_0^\pi \sin\theta\cos^3\frac{\theta}{2}d\theta.$$

ここで，$\theta = 2\varphi$ として，

$$S = 16\pi\int_0^{\frac{\pi}{2}} \sin 2\varphi\cos^3\varphi d\varphi = 32\pi\int_0^{\frac{\pi}{2}}\sin\varphi\cos^4\varphi d\varphi = 32\left[\frac{\cos^5\varphi}{5}\right]_0^{\frac{\pi}{2}} = \frac{32}{5}\pi.$$

(4)

$$S = 2\pi\int_0^{\frac{\pi}{4}} \tan x\sqrt{1+\frac{1}{\cos^4 x}}dx = 2\pi\int_0^{\frac{\pi}{4}}\frac{\sin x\sqrt{1+\cos^4 x}}{\cos^3 x}dx.$$

ここで，$t=\cos x$ と変換して，$S = 2\pi\displaystyle\int_{\frac{1}{\sqrt{2}}}^1 \frac{\sqrt{t^4+1}}{t^3}dt$. さらに，$t = u^{\frac{1}{4}}$ とすると，$\dfrac{dt}{du} = \dfrac{1}{4}u^{-\frac{3}{4}}$ なので，

$$\begin{aligned} S &= \frac{\pi}{2}\int_{\frac{1}{4}}^1 u^{-\frac{3}{2}}\sqrt{u+1}du = \frac{\pi}{2}\left(\left[-2u^{-\frac{1}{2}}\sqrt{u+1}\right]_{\frac{1}{4}}^1 + \int_{\frac{1}{4}}^1\frac{du}{\sqrt{u^2+u}}\right) \\ &= \pi\left(\sqrt{5}-\sqrt{2}\right) + \frac{\pi}{2}\left[2\log\left(\sqrt{x}+\sqrt{x+1}\right)\right]_{\frac{1}{4}}^1 = \pi\left\{\sqrt{5}-\sqrt{2}+\log\frac{2\left(1+\sqrt{2}\right)}{1+\sqrt{5}}\right\}. \end{aligned}$$

(5)　対称性より，$z \geqq 0$ としてよい．

$$z_x = x\frac{1-2\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}-x^2-y^2}},\quad z_y = y\frac{1-2\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}-x^2-y^2}}$$

なので，$\sqrt{1+z_x^2+z_y^2} = \dfrac{1}{2\sqrt{\sqrt{x^2+y^2}-x^2y^2}}$. $D\colon x^2+y^2 \leqq 1$ とし，極座標変換より，

$$\begin{aligned} S &= \iint_D \frac{dxdy}{\sqrt{\sqrt{x^2+y^2}-x^2-y^2}} = \int_0^1 dr\int_0^{2\pi}\frac{r}{\sqrt{r-r^2}}d\theta \\ &= 2\pi\int_0^1 r^{\frac{1}{2}}(1-r)^{-\frac{1}{2}}dr = 2\pi B\left(\frac{3}{2},\frac{1}{2}\right) = 2\pi\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \pi^2. \end{aligned}$$

6.　全質量を M とし，求める重心を $G(x_0,y_0,z_0)$，慣性モーメントを I とする．

(1)

$$M = \iiint_U |xyz|dxdydz = 4\left(\int_0^1 xdx\right)^3 = \frac{1}{2}.$$

また，

$$x_0 = \frac{1}{M}\iiint_U x|xyz|dxdydz = 8\left(\int_0^1 x^2dx\right)\left(\int_0^1 ydy\right)^2 = \frac{2}{3}.$$

対称性より，$y_0 = z_0 = 0$. よって，$G\left(\dfrac{2}{3},0,0\right)$.

$$I = \iiint_U (x^2+z^2)|xyz|dxdydz$$

$$
\begin{aligned}
&= 4\left(\int_0^1 x^3dx\right)\left(\int_0^1 ydy\right)^2 + 4\left(\int_0^1 xdx\right)^2\left(\int_0^1 z^3dy\right)\\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}
$$

(2)　円柱座標変換より，

$$
M = \iiint_U \sqrt{x^2+y^2}dxdydz = \left(\int_0^1 dz\right)\left(\int_0^1 dr\int_0^{2\pi} r^2 d\theta\right) = 2\pi\left[\frac{r^3}{3}\right]_0^1 = \frac{2}{3}\pi.
$$

対称性より，$x_0 = y_0 = 0$.

$$
z_0 = \frac{1}{M}\iiint_U z\sqrt{x^2+y^2}dxdydz = \frac{3}{2\pi}\left(\int_0^1 zdz\right)\left(\int_0^1 dr\int_0^{2\pi} r^2d\theta\right) = 3\left[\frac{z^2}{2}\right]_0^1\left[\frac{r^3}{3}\right]_0^1 = \frac{1}{2}.
$$

よって，$G\left(0, 0, \dfrac{1}{2}\right)$.

$$
I = \iiint_U (x^2+y^2)^{\frac{3}{2}}dxdydz = \left(\int_0^1 dz\right)\left(\int_0^1 dr\int_0^{2\pi} r^4d\theta\right) = 2\pi\left[\frac{z^2}{2}\right]_0^1\left[\frac{r^5}{5}\right]_0^1 = \frac{2}{5}\pi.
$$

(3)

$$
\begin{aligned}
M &= \iiint_U zdxdydz = \int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y} zdz\\
&= \frac{1}{2}\int_0^1 dx\int_0^{1-x}(1-x-y)^2dy = \frac{1}{2}\int_0^1\left[-\frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x}dx\\
&= \frac{1}{6}\int_0^1(1-x)^3dx = \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}.
\end{aligned}
$$

同様の計算で，

$$
x_0 = \frac{1}{M}\iiint_U xzdxdydz = 4\int_0^1 x(1-x)^3dx = 4\left(\left[-x\frac{(1-x)^4}{4}\right]_0^1 + \int_0^1\frac{(1-x)^4}{4}dx\right) = \frac{1}{5}.
$$

対称性より，$y_0 = \dfrac{1}{5}$. また，

$$
\begin{aligned}
z_0 &= \frac{1}{M}\iiint_U z^2dxdydz = 24\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y} z^2dz\\
&= 8\int_0^1 dx\int_0^{1-x}(1-x-y)^3dy = 2\int_0^1(1-x)^4dx = \frac{2}{5}.
\end{aligned}
$$

よって，$G\left(\dfrac{1}{5}, \dfrac{1}{5}, \dfrac{2}{5}\right)$.

$$
\begin{aligned}
I &= \iiint_U (x^2+z^2)zdxdydz = \int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}(x^2z+z^3)dz\\
&= \int_0^1 dx\int_0^{1-x}\left\{x^2\frac{(1-x-y)^2}{2} + \frac{(1-x-y)^4}{4}\right\}dy\\
&= \int_0^1\left\{x^2\frac{(1-x)^3}{6} + \frac{(1-x)^5}{20}\right\}dx
\end{aligned}
$$

$$= \frac{1}{6}\cdot\frac{2}{4\cdot5\cdot6} + \frac{1}{20}\frac{1}{6} = \frac{1}{120}\left(\frac{1}{3}+1\right) = \frac{1}{90}.$$

(4)　極座標変換より，

$$\begin{aligned} M &= \iiint_U ydxdydz = \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi} r^3 \sin^2\theta \sin\varphi d\varphi \\ &= \left(\int_0^1 r^3 dr\right)\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta\right)\left(\int_0^{\pi} \sin\varphi d\varphi\right) = \frac{1}{4}\frac{\pi}{4}2 = \frac{\pi}{8}. \end{aligned}$$

対称性より，$x_0 = 0$. また，

$$\begin{aligned} y_0 &= \frac{1}{M}\iiint_U y^2 dxdydz = \frac{8}{\pi}\left(\int_0^1 r^4 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta\right)\left(\int_0^{\pi} \sin^2\varphi d\varphi\right) \\ &= \frac{8}{\pi}\frac{1}{5}\left(\int_0^{\frac{\pi}{2}} \frac{3\sin\theta - \sin 3\theta}{4} d\theta\right)\left(\int_0^{\pi} \frac{1-\cos 2\varphi}{2} d\varphi\right) \\ &= \frac{8}{\pi}\frac{1}{5}\cdot\frac{1}{4}\left(3-\frac{1}{3}\right)\frac{\pi}{2} = \frac{8}{15}, \\ z_0 &= \frac{1}{M}\iiint_U zydxdydz = \frac{8}{\pi}\left(\int_0^1 r^4 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta d\theta\right)\left(\int_0^{\pi} \sin\varphi d\varphi\right) \\ &= \frac{8}{\pi}\frac{1}{5}\left[\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}}\cdot 2 = \frac{16}{15\pi}. \end{aligned}$$

よって，$G\left(0, \frac{8}{15}, \frac{16}{15\pi}\right)$.

$$\begin{aligned} I &= \iiint_U (y^2+z^2)ydxdydz \\ &= \left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi} \sin^2\theta\sin\varphi(\sin^2\theta\sin^2\varphi + \cos^2\theta)d\varphi\right) \\ &= \frac{1}{6}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi} \sin^2\theta\sin\varphi(\sin^2\varphi + \cos^2\theta\cos^2\varphi)d\varphi \\ &= \frac{1}{6}\left\{\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta\right)\left(\int_0^{\pi} \frac{3\sin\varphi - \sin 3\varphi}{4} d\varphi\right) + \left(\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4} d\theta\right)\left[-\frac{\cos^3\varphi}{3}\right]_0^{\pi}\right\} \\ &= \frac{1}{6}\left(\frac{\pi}{4}\cdot\frac{4}{3} + \frac{\pi}{16}\cdot\frac{2}{3}\right) = \frac{\pi}{16}. \end{aligned}$$

(5)　極座標変換より，

$$\begin{aligned} M &= \iiint_U z^2 dxdydz = \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} r^4 \sin\theta\cos^2\theta d\theta \\ &= \left(\int_0^1 r^4 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} d\varphi\right) \\ &= \frac{1}{5}\cdot\left[-\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2} = \frac{\pi}{30}. \end{aligned}$$

$$\begin{aligned} x_0 &= \frac{1}{M}\iiint_U xz^2 dxdydz = \frac{30}{\pi}\left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos\varphi d\varphi\right) \\ &= \frac{30}{\pi}\cdot\frac{1}{6}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{8} d\theta\right) = \frac{5}{\pi}\cdot\frac{\pi}{16} = \frac{5}{16}, \end{aligned}$$

$$z_0 = \frac{1}{M}\iiint_U z^3 dxdydz = \frac{30}{\pi}\left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin\theta\cos^3\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} d\varphi\right)$$
$$= \frac{30}{\pi}\cdot\frac{1}{6}\left[-\frac{\cos^4\theta}{4}\right]_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2} = \frac{5}{8}.$$

対称性より，$y_0 = x_0 = \dfrac{5}{16}$．よって，$G\left(\dfrac{5}{16}, \dfrac{5}{16}, \dfrac{5}{8}\right)$.

$$I = \iiint_U (y^2+z^2)z^2 dxdydz = \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} r^6(\sin^2\theta\sin^2\varphi + \cos^2\theta)\sin\theta\cos^2\theta d\varphi$$
$$= \left(\int_0^1 r^6 dr\right)\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} (\sin^2\varphi + \cos^2\theta\cos^2\varphi)\sin\theta\cos^2\theta d\varphi$$
$$= \frac{1}{7}\left\{\left(\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi\right) + \left(\int_0^{\frac{\pi}{2}} \sin\theta\cos^4\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos^2\varphi d\varphi\right)\right\}$$
$$= \frac{1}{7}\left\{\left[-\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\varphi}{2} d\varphi\right) + \left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}} \frac{\cos 2\varphi + 1}{2} d\varphi\right)\right\}$$
$$= \frac{1}{7}\left(\frac{1}{3}\cdot\frac{\pi}{4} + \frac{1}{5}\cdot\frac{\pi}{4}\right) = \frac{2}{105}\pi.$$