演習問題 5

1. 求める面積を A とおく.

(1)　$2\left(x+y+\frac{1}{4}\right)^2+y^2=\frac{9}{8}$ が囲む面積を求めればよい．$u=x+y+\frac{1}{4}$，$v=y$ により，$E\colon 2u^2+v^2\leqq\frac{9}{8}$, $u\geqq 0$, $v\geqq 0$ は $D\colon 2x^2+3y^2+4xy+x+y\leqq 1$, $x+y+\frac{1}{4}\geqq 0$, $y\geqq 0$ に写り，$\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix}1&1\\0&1\end{pmatrix}=1$．対称性より，

$$\begin{aligned}A&=4\iint_D dxdy=4\int_0^{\frac{3}{4}}du\int_0^{\sqrt{\frac{9}{8}-2u^2}}dv=4\int_0^{\frac{3}{4}}\sqrt{\frac{9}{8}-2u^2}du\\&=4\left[\frac{4x\sqrt{9-16x^2}+9\sin^{-1}\frac{4}{3}x}{16\sqrt{2}}\right]_0^{\frac{3}{4}}=\frac{9}{8\sqrt{2}}\pi.\end{aligned}$$

(2)　$x^2+2\sqrt{3}xy-y^2=-2$ は $\left(\frac{x-\sqrt{3}y}{2}\right)^2-\left(\frac{\sqrt{3}x+y}{2}\right)^2=1$ とかける．$u=\frac{x-\sqrt{3}y}{2}$, $v=\frac{\sqrt{3}x+y}{2}$ により，$E\colon u^2-v^2\leqq 1$, $u\leqq 2$ は $D\colon x^2+2\sqrt{3}xy-y^2\leqq -2$, $y\geqq\frac{x}{\sqrt{3}}-\frac{4}{\sqrt{3}}$ に写り，$\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}=1$ なので，対称性より，

$$\begin{aligned}A&=\iint_D dxdy=2\int_1^2du\int_0^{\sqrt{u^2+1}}dv=2\int_1^2\sqrt{u^2+1}du\\&=\left[x\sqrt{x^2-1}-\log\left(\sqrt{x^2-1}+x\right)\right]_1^2=2\sqrt{3}-\log\left(2+\sqrt{3}\right).\end{aligned}$$

(3)　$x=r\cos\theta$，$y=r\sin\theta$ により，$E\colon 0\leqq r\leqq\sqrt{\frac{\sin 2\theta}{2}}$, $0\leqq\theta\leqq\frac{\pi}{4}$ は $D\colon (x^2+y^2)^2\leqq xy$, $x\geqq y\geqq 0$ に写るので，対称性より，

$$A=4\iint_D dxdy=4\int_0^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{\frac{\sin 2\theta}{2}}}rdr=\int_0^{\frac{\pi}{4}}\sin 2\theta d\theta=\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{1}{2}.$$

(4)　$D\colon \sqrt[4]{x}+\sqrt[4]{y}\leqq 1$, $\sqrt[4]{x-1}+\sqrt[4]{y}\leqq 1$, $x\geqq 0$, $y\geqq 0$ とおくと，対称性より，

$$\begin{aligned}A&=4\iint_D dxdy=4\int_0^{\frac{1}{2}}\left(1-\sqrt[4]{1-x}\right)^4dx\\&=4\int_0^{\frac{1}{2}}\left(1-4\sqrt[4]{1-x}+6\sqrt{1-x}-4\sqrt[4]{1-x}^3+1-x\right)dx\\&=\left[2x-\frac{x^2}{2}+\frac{16}{5}(1-x)^{\frac{5}{4}}-4(1-x)^{\frac{3}{2}}+\frac{16}{7}(1-x)^{\frac{7}{4}}\right]_0^{\frac{1}{2}}\\&=-\frac{171}{70}-4\sqrt{2}+\frac{16}{7}\sqrt[4]{2}+\frac{16}{5}\sqrt[4]{8}.\end{aligned}$$

(5)　$x=r\cos\theta$，$y=r\sin\theta$ により，$E\colon 0\leqq r\leqq\frac{3\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$, $0\leqq\theta\leqq\frac{\pi}{4}$ は $D\colon x^3+y^3\leqq 3xy$, $x\geqq y\geqq 0$ に写るので，対称性より，

$$A=2\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{3\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}}rdr=9\int_0^{\frac{\pi}{4}}\frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}d\theta$$

$$= 9\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta\cos^2\theta}{\sin^6\theta+\cos^6\theta+2\sin^3\theta\cos^3\theta}d\theta.$$

ここで，$t=\tan\theta$ とおくと，

$$A = 9\int_0^1 \frac{t^2}{1+t^6+2t^3}dt = 9\int_0^1 \frac{t^2}{(t^3+1)^2}dt = 3\left[-\frac{1}{t^3+1}\right]_0^1 = \frac{3}{2}.$$

(6)　$D\colon 0 \leqq y \leqq 1,\ 0 \leqq x \leqq \sqrt{y-y^4}$ とおくと，対称性より，$A = 2\int_0^1 \sqrt{y-y^4}dy$. ここで，$t = y^3$ とすると $\dfrac{dy}{dt} = \dfrac{1}{3}t^{-\frac{2}{3}}$ より，$A = \dfrac{2}{3}\displaystyle\int_0^1 \sqrt{\frac{1-t}{t}}dt$. また，$u = \sqrt{\dfrac{1-t}{t}}$ とすると $\dfrac{dt}{du} = -\dfrac{2u}{u^2+1}$ なので，

$$\begin{aligned} A &= \frac{4}{3}\int_0^\infty \frac{u^2}{(u^2+1)^2}du = \frac{4}{3}\int_0^\infty \left(\frac{1}{u^2+1}-\frac{1}{(u^2+1)^2}\right)du \\ &= \frac{2}{3}\lim_{n\to\infty}\left[-\frac{u}{u^2+1}+\tan^{-1}u\right]_0^n = \frac{\pi}{3}. \end{aligned}$$

(7)　$x = r\cos\theta$，$y = r\sin\theta$ とすると，対称性より，

$$\begin{aligned} A &= 4\int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{\frac{\sin\theta\cos\theta}{\cos^4\theta+\sin^4\theta}}} rdr = 2\int_0^{\frac{\pi}{4}} \frac{\sin\theta\cos\theta}{\cos^4\theta+\sin^4\theta}d\theta = 2\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta}{\cos^2 2\theta+1}d\theta \\ &= \left[-\tan^{-1}\cos 2\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}. \end{aligned}$$

2.

(1)　$x = \dfrac{r\cos\theta-1}{2}$，$y = \dfrac{r\sin\theta+2}{3}$ により，$E\colon 0 \leqq r \leqq \sqrt{5},\ 0 \leqq \theta \leqq 2\pi$ は $D\colon 4\left(x+\dfrac{1}{2}\right)^2 + 9\left(y-\dfrac{2}{3}\right)^2 \leqq 5$ に写り，$\dfrac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\cos\theta}{2} & -\frac{r}{2}\sin\theta \\ \frac{\sin\theta}{3} & \frac{r}{3}\cos\theta \end{pmatrix} = \dfrac{r}{6}$ なので，

$$\begin{aligned} \iint_D xydxdy &= \frac{1}{36}\iint_E r(r\cos\theta-1)(r\sin\theta+2)drd\theta \\ &= \frac{1}{36}\int_0^{\sqrt{5}} dr \int_0^{2\pi} r\left\{r^2\frac{\sin 2\theta}{2} + r(2\cos\theta-\sin\theta) - 2\right\}d\theta \\ &= \frac{1}{36}\int_0^{\sqrt{5}} r\left[-r^2\frac{\cos 2\theta}{2} + r(2\sin\theta+\cos\theta) - 2\theta\right]_{\theta=0}^{\theta=2\pi} dr = -\frac{\pi}{9}\int_0^{\sqrt{5}} rdr = -\frac{5}{18}\pi. \end{aligned}$$

(2)　$D_n\colon \dfrac{1}{n} \leqq |x| \leqq 1,\ |x|+|y| \leqq 1$ とすると，対称性より，

$$\begin{aligned} &\iint_D \frac{dxdy}{\sqrt{x^2+y^2}}dxdy = \lim_{n\to\infty}\iint_{D_n} \frac{dxdy}{\sqrt{x^2+y^2}} = 4\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx \int_0^{1-x} \frac{dy}{\sqrt{x^2+y^2}} \\ &= 4\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left[\log\left(y+\sqrt{x^2+y^2}\right)\right]_{y=0}^{y=1-x} dx \\ &= 4\lim_{n\to\infty}\int_{\frac{1}{n}}^1 \left\{\log\left(1-x+\sqrt{2x^2-2x+1}\right) - \log x\right\}dx \\ &= 4\lim_{n\to\infty}\left\{\left[x\left\{\log\left(1-x+\sqrt{2x^2-2x+1}\right) - \log x\right\}\right]_{\frac{1}{n}}^1 + \int_{\frac{1}{n}}^1 \frac{dx}{\sqrt{2x^2-2x+1}}\right\} \end{aligned}$$

$$= 4\left[-\frac{1}{\sqrt{2}}\log\left|1-2x+\sqrt{4x^2-4x+2}\right|\right]_0^1 = 4\sqrt{2}\log\left(1+\sqrt{2}\right).$$

(3)　$D_n\colon 0 \leqq x \leqq n,\ 0 \leqq y \leqq n$ とすると，対称性より，

$$\iint_D \frac{dxdy}{e^x+e^y} = \lim_{n\to\infty}\iint_{D_n}\frac{dxdy}{e^x+e^y} = 2\lim_{n\to\infty}\int_0^n dx\int_0^x \frac{dy}{e^x+e^y}$$
$$= 2\lim_{n\to\infty}\int_0^n\left[\frac{y-\log(e^x+e^y)}{e^x}\right]_{y=0}^{y=x}dx = 2\lim_{n\to\infty}\int_0^n e^{-x}\left\{-\log 2+\log(e^x+1)\right\}dx$$
$$= 2\lim_{n\to\infty}\left\{\left[-e^{-x}\left\{-\log 2+\log(e^x+1)\right\}\right]_0^n + \int_0^n\frac{dx}{e^x+1}\right\}$$
$$= 2\lim_{n\to\infty}\left[x-\log(e^x+1)\right]_0^n = 2\log 2.$$

(4)　$D_n\colon 1 \leqq x \leqq n,\ 0 \leqq y \leqq \log x$ とすると，

$$\iint_D \frac{y^2}{x^3}dxdy = \lim_{n\to\infty}\int_1^n dx\int_0^{\log x}\frac{y^2}{x^3}dy = \lim_{n\to\infty}\int_1^n\frac{1}{x^3}\left[\frac{y^3}{3}\right]_0^{\log x}dx = \frac{1}{3}\lim_{n\to\infty}\int_1^n\frac{\log^3 x}{x^3}dx.$$

ここで，$x=e^t$ と変換して，$\displaystyle\iint_D\frac{y^2}{x^3}dxdy = \frac{1}{3}\lim_{n\to\infty}\int_0^{\log n}t^3e^{-2t}dt$．さらに，$t=\dfrac{s}{2}$ と変換して，

$$\iint_D\frac{y^2}{x^3}dxdy = \frac{1}{48}\int_0^\infty s^3e^{-s}ds = \frac{1}{48}\Gamma(4) = \frac{1}{8}.$$

(5)　対称性より，

$$\iint_D (x^2+y^2)dxdy = 4\int_0^1 dx\int_0^{(1-\sqrt{x})^2}(x^2+y^2)dy$$
$$= 4\int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=(1-\sqrt{x})^2}dx = 4\int_0^1\left\{x^2(1-\sqrt{x})^2+\frac{(1-\sqrt{x})^6}{3}\right\}dx.$$

ここで，$x=t^2$ と変換して，

$$\iint_D (x^2+y^2)dxdy = \frac{8}{3}\int_0^1\left\{3t^5(1-t)^2+t(1-t)^6\right\}dt$$
$$= \frac{8}{3}\left\{3B(6,3)+B(2,7)\right\} = \frac{8}{3}\left\{3\frac{\Gamma(6)\Gamma(3)}{\Gamma(9)}+\frac{\Gamma(2)\Gamma(7)}{\Gamma(9)}\right\} = \frac{2}{21}.$$

(6)　$D_n\colon x^2+y^2 \leqq n^2$ とし，極座標変換より，

$$\iint_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\sin\sqrt{x^2+y^2}\,dxdy = \lim_{n\to\infty}\iint_{D_n}e^{-\sqrt{x^2+y^2}}\sin\sqrt{x^2+y^2}\,dxdy$$
$$= \lim_{n\to\infty}\int_0^n dr\int_0^{2\pi}re^{-r}\sin r\,d\theta = 2\pi\lim_{n\to\infty}\int_0^n re^{-r}\sin r\,dr.$$

ここで，

$$I = \int e^{-r}\sin r\,dr = -e^{-r}\sin r + \int e^{-r}\cos r = -e^{-r}\sin r - e^{-r}\cos r - I$$

なので，$I = -\dfrac{1}{2}e^{-r}(\sin r+\cos r)$．同様にして，

$$\int e^{-r}\cos r\,dr = \frac{1}{2}e^{-r}(\sin r-\cos r).$$

また,

$$
\begin{aligned}
J &= \int re^{-r}\sin r dr = -re^{-r}\sin r + \int e^{-r}(\sin r + r\cos r) \\
&= -re^{-r}\sin r + I - e^{-r} r\cos r + \int e^{-r}(\cos r - r\sin r)dr \\
&= -re^{-r}(\sin r + \cos r) - e^{-r}\cos r - J
\end{aligned}
$$

より, $J = -\dfrac{1}{2}e^{-r}\{r\sin r + (r+1)\cos r\}$. よって,

$$
\iint_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\sin\sqrt{x^2+y^2}dxdy = -\pi\lim_{n\to\infty}\left[e^{-r}\{r\sin r + (r+1)\cos r\}\right]_0^n = \pi.
$$

(7)　$x = r\cos\theta + 1$, $y = r\sin\theta + 1$ と変換して,

$$
\begin{aligned}
\iint_D (x^5+y^5)dxdy &= \int_0^{2\pi} d\theta \int_0^1 \{r^6(\sin^5\theta + \cos^5\theta) + 5r^5(\sin^4\theta + \cos^4\theta) \\
&\qquad + 10r^4(\sin^3\theta + \cos^3\theta) + 10r^3 + 5r^2(\sin\theta + \cos\theta) + 2r\}dr \\
&= \int_0^{2\pi}\left\{\frac{1}{7}(\sin^5\theta + \cos^5\theta) + \frac{5}{6}(\sin^4\theta + \cos^4\theta) + 2(\sin^3\theta + \cos^3\theta)\right. \\
&\qquad \left. + \frac{5}{2} + \frac{5}{3}(\sin\theta + \cos\theta) + 1\right\}d\theta \\
&= 4\int_0^{\frac{\pi}{2}}\left\{\frac{5}{6}(\sin^4\theta + \cos^4\theta) + \frac{7}{2}\right\}d\theta \\
&= \frac{5}{3}\left\{B\left(\frac{5}{2},\frac{1}{2}\right) + B\left(\frac{1}{2},\frac{5}{2}\right)\right\} + 7\pi = \frac{10}{3}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} + 7\pi = \frac{33}{4}\pi.
\end{aligned}
$$

(8)　極座標変換より,

$$
\begin{aligned}
&\iint_D \frac{(x+y)^2xy}{x^2+y^2}e^{(x+y)^2}dxdy = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta\int_0^1 r^3(1+\sin 2\theta)\sin 2\theta e^{r^2(1+\sin 2\theta)}dr \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\sin 2\theta)\sin 2\theta\left\{\left[\frac{r^2}{2(1+\sin 2\theta)}e^{r^2(1+\sin 2\theta)}\right]_{r=0}^{r=1} - \frac{1}{1+\sin 2\theta}\int_0^1 re^{r^2(1+\sin 2\theta)}dr\right\}d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\sin 2\theta)\sin 2\theta\left\{\frac{1}{2(1+\sin 2\theta)}e^{1+\sin 2\theta}\right. \\
&\qquad \left. - \frac{1}{1+\sin 2\theta}\left[\frac{1}{2(1+\sin 2\theta)}e^{r^2(1+\sin 2\theta)}\right]_{r=0}^{r=1}\right\}d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{1+\sin 2\theta}\left(\sin 2\theta e^{1+\sin 2\theta} + 1\right)d\theta.
\end{aligned}
$$

ここで,

$$
\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{1+\sin 2\theta}d\theta = \int_0^{\frac{\pi}{2}}\left\{1 - \frac{1}{(\sin\theta + \cos\theta)^2}\right\}d\theta = \left[\theta - \frac{\sin\theta}{\sin\theta + \cos\theta}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1
$$

であり,

$$
\int_0^{\frac{\pi}{2}}\frac{\sin^2 2\theta}{1+\sin 2\theta}e^{\sin 2\theta}d\theta = \int_0^{\frac{\pi}{2}}\frac{-\cos^2 2\theta + 1}{(\sin\theta + \cos\theta)^2}e^{\sin 2\theta}d\theta = \int_0^{\frac{\pi}{2}}\frac{\cos 2\theta(\sin^2\theta - \cos^2\theta) + 1}{(\sin\theta + \cos\theta)^2}e^{\sin 2\theta}d\theta
$$

$$= \int_0^{\frac{\pi}{2}} \left\{ 2\cos 2\theta \left(\frac{\sin\theta}{\sin\theta + \cos\theta} - \frac{1}{2} \right) + \frac{1}{(\sin\theta + \cos\theta)^2} \right\} e^{\sin 2\theta} d\theta$$
$$= \left[\left(\frac{\sin\theta}{\sin\theta + \cos\theta} - \frac{1}{2} \right) e^{\sin 2\theta} \right]_0^{\frac{\pi}{2}} = 1$$

より，$\displaystyle\iint_D \frac{(x+y)^2 xy}{x^2+y^2} e^{(x+y)^2} dxdy = \frac{1}{8}(2e + \pi - 2)$.

(9)　$x + y = r\cos\theta$，$x - 2y = r\sin\theta$ とすると，$x = \dfrac{r}{3}(2\cos\theta + \sin\theta)$，$y = \dfrac{r}{3}(\cos\theta - \sin\theta)$ であり，$E\colon 0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq \dfrac{\pi}{2}$ は $D\colon (x+y)^2 + (x-2y)^2 \leqq 1,\ x + y \geqq 0,\ x - 2y \geqq 0$ に写り，

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} \frac{2\cos\theta + \sin\theta}{3} & \frac{r}{3}(-2\sin\theta + \cos\theta) \\ \frac{\cos\theta - \sin\theta}{3} & \frac{r}{3}(-\sin\theta - \cos\theta) \end{pmatrix} = -\frac{r}{3}$$

なので，

$$\iint_D (x+y)\sin(x-2y) dxdy = \frac{1}{3}\iint_E r^2 \cos\theta \sin(r\sin\theta) drd\theta = \frac{1}{3}\int_0^1 r^2 \left[-\frac{\cos(r\sin\theta)}{r} \right]_{\theta=0}^{\theta=\frac{\pi}{2}} dr$$
$$= \frac{1}{3}\int_0^1 r(1-\cos r) dr = \frac{1}{3}\left(\left[\frac{r^2}{2} \right]_0^1 - [r\sin r]_0^1 + \int_0^1 \sin r dr \right) = \frac{1}{3}\left(\frac{1}{2} - \sin 1 + [-\cos r]_0^1 \right)$$
$$= \frac{1}{2} - \frac{\sin 1 + \cos 1}{3}.$$

3.　与えられた集合を D とし，求める面積を V とする．

(1)　対称性より，

$$V = 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} dy \int_0^{(1-\sqrt{y}-\sqrt{z})^2} dx = 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} (1 - \sqrt{y} - \sqrt{z})^2 dy$$
$$= 8\int_0^1 dz \int_0^{(1-\sqrt{z})^2} \left\{ y - 2(1-\sqrt{z})\sqrt{y} + (1-\sqrt{z})^2 \right\} dy$$
$$= 8\int_0^1 \left[\frac{y^2}{2} - \frac{4}{3}(1-\sqrt{z})y^{\frac{3}{2}} + (1-\sqrt{z})^2 y \right]_{y=0}^{y=(1-\sqrt{z})^2} dz = \frac{4}{3}\int_0^1 (1-\sqrt{z})^4 dz.$$

ここで，$z = t^2$ とすると $\dfrac{dz}{dt} = 2t$ なので，

$$V = \frac{8}{3}\int_0^1 t(1-t)^4 dt = \frac{8}{3}\int_0^1 \left\{ (1-t)^4 - (1-t)^5 \right\} dt = \frac{8}{3}\left[-\frac{1}{5}(1-t)^5 + \frac{1}{6}(1-t)^6 \right]_0^1 = \frac{4}{45}.$$

(2)　$|z| = \sqrt{1 - (x^2+y^2)^2}$ なので，$x = r\cos\theta$，$y = r\sin\theta$ として，対称性より，

$$V = 8\int_0^1 dr \int_0^{\frac{\pi}{2}} \sqrt{1-r^4}\, r d\theta = 4\pi \int_0^1 r\sqrt{1-r^4} dr.$$

ここで，$r = t^{\frac{1}{4}}$ とすると，$\dfrac{dr}{dt} = \dfrac{1}{4}t^{-\frac{3}{4}}$ より，

$$V = \pi \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{1}{2}} dt = \pi B\left(\frac{1}{2}, \frac{3}{2} \right) = \pi \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{\pi^2}{2}.$$

(3)　$A(z)\colon x^2 + y^2 \leqq \left(1 - z^{\frac{2}{3}}\right)^3$ とおき，$x = r\cos\theta$，$y = r\sin\theta$ とすると，

$$V = \int_{-1}^1 dz \iint_{A(z)} dxdy = 2\left(\int_0^1 dz \int_0^{(1-z^{\frac{2}{3}})^{\frac{3}{2}}} rdr \right)\left(\int_0^{2\pi} d\theta \right) = 2\pi \int_0^1 \left(1 - z^{\frac{2}{3}}\right)^3 dz$$

$$= 2\pi \int_0^1 \left(1 - 3z^{\frac{2}{3}} + 3z^{\frac{4}{3}} - z^2\right) dz = 2\pi \left[z - \frac{9}{5}z^{\frac{5}{3}} + \frac{9}{7}z^{\frac{7}{3}} - \frac{z^3}{3}\right]_0^1 = \frac{32}{105}\pi.$$

(4)　$x = r\cos\theta$，$y = r\sin\theta$ とすると，$0 \leqq \theta \leqq \pi$ で，$r = \cos\theta + 1$ なので，$x = \cos\theta(\cos\theta + 1)$，$y = \sin\theta(\cos\theta + 1)$．$x' = \dfrac{dx}{d\theta} = -\sin\theta(1 + 2\cos\theta)$ より，

$$\begin{aligned}
V &= \pi \int_0^{\pi} y^2 x' d\theta = -\pi \int_0^{\pi} \sin^3\theta(\cos\theta + 1)^2(2\cos\theta + 1)d\theta \\
&= -\pi \int_0^{\pi} \sin^3\theta(2\cos^3\theta + 5\cos^2\theta + 4\cos\theta + 1)d\theta \\
&= \pi \int_0^{\frac{\pi}{2}} \left(10\sin^3\theta\cos^2\theta + 2\sin^3\theta\right) d\theta \\
&= \pi \left\{5B\left(2, \frac{3}{2}\right) + B\left(2, \frac{1}{2}\right)\right\} = \pi \left\{5\frac{\Gamma(2)\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} + \frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})}\right\} = \frac{8}{3}\pi.
\end{aligned}$$

(5)　$x = r\sin\theta\cos\varphi$，$y = r\sin\theta\sin\varphi$，$z = r\cos\theta$ とすると，$r = \sin^2\theta\cos\theta\sin\varphi\cos\varphi$ が囲む集合の体積を求めればよいので，対称性より，

$$\begin{aligned}
V &= 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sin^2\theta\cos\theta\sin\varphi\cos\varphi} r^2 \sin\theta dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^3\theta d\theta \int_0^{\frac{\pi}{2}} \sin^3\varphi\cos^3\varphi d\varphi \\
&= \frac{1}{3}B(4,2)B(2,2) = \frac{1}{3}\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)}\frac{\Gamma(2)^2}{\Gamma(4)} = \frac{1}{360}.
\end{aligned}$$

(6)　$A(r) = \left\{(x, y) \;\middle|\; x^{\frac{2}{3}} + y^{\frac{2}{3}} \leqq r^{\frac{2}{3}}\right\}$ $(r > 0)$ の面積は，対称性より，

$$|A(r)| = 4\int_0^r \left(r^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx.$$

ここで，$x = ru^{\frac{3}{2}}$ とすると，$\dfrac{dx}{du} = \dfrac{3}{2}ru^{\frac{1}{2}}$ なので，

$$|A(r)| = 6r^2 \int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{3}{2}}du = 6r^2 B\left(\frac{3}{2}, \frac{5}{2}\right) = 6r^2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{3}{8}\pi r^2.$$

対称性より，

$$V = \iiint_D dxdydz = 2\int_0^1 A\left((1 - z^{\frac{2}{3}})^{\frac{3}{2}}\right) dz = \frac{3}{4}\pi \int_0^1 \left(1 - z^{\frac{2}{3}}\right)^3 dz.$$

また，$z = t^{\frac{3}{2}}$ とすると，$\dfrac{dz}{dt} = \dfrac{3}{2}t^{\frac{1}{2}}$ より，

$$V = \frac{9}{8}\pi \int_0^1 t^{\frac{1}{2}}(1-t)^3 dt = \frac{9}{8}\pi B\left(\frac{3}{2}, 4\right) = \frac{9}{8}\pi\frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(\frac{11}{2})} = \frac{4}{35}\pi.$$

(7)　デカルトの正葉線なので，$x = \dfrac{3t}{1+t^3}$，$y = \dfrac{3t^2}{1+t^3}$ $(0 \leqq t < \infty)$ とかける．$x' = \dfrac{3(t - 2t^3)}{(1+t^3)^2}$ なので，変数変換公式より，

$$V = \pi \int_0^{\infty} \left(\frac{3t^2}{1+t^3}\right)^2 \frac{3(2t^3 - 1)}{(1+t^3)^2} dt = 27\pi \int_0^{\infty} \frac{t^4(2t^3-1)}{(t^3+1)^4} dt.$$

ここで，$t=u^{\frac{1}{3}}$ とすると，$\dfrac{dt}{du}=\dfrac{1}{3}u^{-\frac{2}{3}}$ なので，$V=9\pi\displaystyle\int_0^\infty \frac{u^{\frac{2}{3}}(2u-1)}{(u+1)^4}du$．さらに，$u=\dfrac{1-v}{v}$ とすると，$\dfrac{du}{dv}=-\dfrac{1}{v^2}$ なので，

$$
\begin{aligned}
V &= 9\pi\int_0^1\left(\frac{1-v}{v}\right)^{\frac{2}{3}}\frac{2-3v}{v}v^4\frac{1}{v^2}dv = 9\pi\left\{2B\left(\frac{4}{3},\frac{5}{3}\right)-3B\left(\frac{7}{3},\frac{5}{3}\right)\right\} \\
&= \frac{2}{3}\pi\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{3}\pi B\left(\frac{1}{3},\frac{2}{3}\right) = \frac{2}{3}\pi\int_0^1 v^{-\frac{2}{3}}(1-v)^{-\frac{1}{3}}dv \\
&= \frac{2}{3}\pi\int_0^\infty \frac{u^{-\frac{1}{3}}}{1+u}du = 2\pi\int_0^\infty\frac{t}{t^3+1}dt = 2\pi\lim_{n\to\infty}\int_0^n\left\{\frac{t+1}{3(t^2-t+1)}-\frac{1}{3(t+1)}\right\}dt \\
&= 2\pi\lim_{n\to\infty}\int_0^n\left\{\frac{2t-1}{6(t^2-t+1)}+\frac{1}{2(t-\frac{1}{2})^2+\frac{3}{2}}-\frac{1}{3(t+1)}\right\}dt \\
&= 2\pi\lim_{n\to\infty}\left[\frac{1}{6}\log\frac{t^2-t+1}{t^2+2t+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2t-1}{\sqrt{3}}\right]_0^n = \frac{4\sqrt{3}}{9}\pi^2.
\end{aligned}
$$

4.

(1)　対称性より，

$$
\begin{aligned}
&\iiint_D\frac{dxdydz}{(|x|+|y|+|z|+1)^3} = 8\int_0^1 dz\int_0^{1-z}dy\int_0^{1-y-z}\frac{dx}{(x+y+z+1)^3} \\
&= 8\int_0^1 dz\int_0^{1-z}\left[-\frac{1}{2(x+y+z+1)^2}\right]_{z=0}^{z=1-x-y}dy = 4\int_0^1 dz\int_0^{1-z}\left\{\frac{1}{(x+y+1)^2}-\frac{1}{4}\right\}dy \\
&= 4\int_0^1\left[-\frac{1}{x+y+1}-\frac{y}{4}\right]_{y=0}^{y=1-z}dz = 4\int_0^1\left(\frac{1}{z+1}+\frac{z}{4}-\frac{3}{4}\right)dz \\
&= 4\left[\log(z+1)+\frac{z^2}{8}-\frac{3}{4}z\right]_0^1 = 4\log 2-\frac{5}{2}.
\end{aligned}
$$

(2)　$D_n\colon x^2+y^2+z^2\leqq\sqrt{1-\dfrac{1}{n}}$ とすると，対称性と極座標変換より，

$$
\begin{aligned}
&\iiint_D\frac{|z|}{\sqrt{1-x^2-y^2-z^2}}dxdydz = \lim_{n\to\infty}\iiint_{D_n}\frac{|z|}{\sqrt{1-x^2-y^2-z^2}}dxdydz \\
&= 8\lim_{n\to\infty}\int_0^{1-\frac{1}{n}}dr\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\frac{r^3\sin\theta\cos\theta}{\sqrt{1-r^2}}d\varphi = 2\pi\lim_{n\to\infty}\left(\int_0^{1-\frac{1}{n}}\frac{r^3}{\sqrt{1-r^2}}dr\right)\left(\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta\right) \\
&= 2\pi\lim_{n\to\infty}\left(\left[-r^2\sqrt{1-r^2}\right]_0^{1-\frac{1}{n}}+2\int_0^{1-\frac{1}{n}}r\sqrt{1-r^2}dr\right)\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} \\
&= 4\pi\left[-\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{8}{3}\pi.
\end{aligned}
$$

(3)　$D_n\colon \dfrac{1}{n}\leqq|x|\leqq 1,\ \dfrac{1}{n}\leqq|y|\leqq 1,\ \dfrac{1}{n}\leqq|z|\leqq 1$ とすると，対称性より，

$$
\begin{aligned}
&\iiint_D\log|xyz|dxdydz = \lim_{n\to\infty}\iiint_{D_n}\log|xyz|dxdydz = 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1 dy\int_{\frac{1}{n}}^1\log(xyz)dz \\
&= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1\left[\log(xy)z+z(\log z-1)\right]_{z=\frac{1}{n}}^{z=1}dy = 8\lim_{n\to\infty}\int_{\frac{1}{n}}^1 dx\int_{\frac{1}{n}}^1\{\log(xy)-1\}dy
\end{aligned}
$$

$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^{1}\left[(\log x-1)\,y+y\,(\log y-1)\right]_{y=\frac{1}{n}}^{y=1}dx$$
$$= 8\lim_{n\to\infty}\int_{\frac{1}{n}}^{1}(\log x-2)\,dx = 8\lim_{n\to\infty}\left[x\,(\log x-1)-2x\right]_{\frac{1}{n}}^{1} = -24.$$

(4)　対称性より，

$$\iiint_D x^2y^2z^2dxdydz = 8\int_0^1 dy\int_0^y dz\int_0^{y-z}x^2y^2z^2dx$$
$$= 8\int_0^1 dy\int_0^y y^2z^2\left[\frac{x^3}{3}\right]_{x=0}^{x=y-z}dy = \frac{8}{3}\int_0^1 dy\int_0^y (y-z)^3y^2z^2dz.$$

ここで，$z = y\tilde{z}$ とすると，

$$\iiint_D x^2y^2z^2dxdydz = \frac{8}{3}\int_0^1 dy\int_0^1 \tilde{z}^2(1-\tilde{z})^3y^8d\tilde{z}$$
$$= \frac{8}{3}\left(\int_0^1 y^8dy\right)\left(\int_0^1\left(\tilde{z}^2-3\tilde{z}^3+3\tilde{z}^4-\tilde{z}^5\right)d\tilde{z}\right)$$
$$= \frac{8}{3}\left[\frac{y^9}{9}\right]_0^1\left[\frac{\tilde{z}^3}{3}-\frac{3}{4}\tilde{z}^4+\frac{3}{5}\tilde{z}^5-\frac{\tilde{z}^6}{6}\right]_0^1 = \frac{2}{405}.$$

(5)　D_n: $\dfrac{1}{n^2} \leqq x^2+y^2+\dfrac{z^2}{9} \leqq 1$ とし，$x = r\sin\theta\cos\varphi$，$y = r\sin\theta\sin\varphi$，$z = 3r\cos\theta$ とすると，対称性より，

$$\iiint_D \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \lim_{n\to\infty}\iiint_{D_n}\frac{dxdydz}{\sqrt{x^2+y^2+z^2}}$$
$$= 24\lim_{n\to\infty}\int_0^1 dr\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\frac{r\sin\theta}{\sqrt{1+8\cos^2\theta}}d\varphi$$
$$= 12\pi\left(\int_0^1 rdr\right)\left(\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta\right) = 6\pi\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta.$$

ここで，$t = \cos\theta$ と置換すると，

$$\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}d\theta = \int_0^1\frac{dt}{\sqrt{1+8t^2}} = \left[\frac{\sinh^{-1}(2\sqrt{2}t)}{2\sqrt{2}}\right]_0^1 = \frac{\sinh^{-1}(2\sqrt{2})}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$$

なので，$\displaystyle\iiint_D \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = 3\sqrt{2}\pi\log\left(1+\sqrt{2}\right).$

(6)　$x = r\sin\theta\cos\varphi$，$y = r\cos\theta$，$z = r\sin\theta\sin\varphi$ とすると，

$$\iiint_D \frac{dxdydz}{\sqrt{x^2+(y-1)^2+z^2}} = \int_0^2 dr\int_0^{\pi}d\theta\int_0^{2\pi}\frac{r^2\sin\theta}{\sqrt{r^2-2r\cos\theta+1}}d\varphi$$
$$= 2\pi\int_0^2 r^2\left[\frac{1}{r}\sqrt{r^2-2r\cos\theta+1}\right]_{\theta=0}^{\theta=\pi}dr = 2\pi\int_0^2 r\,(r+1-|r-1|)\,dr$$
$$= 2\pi\left(\int_0^1 2r^2dr+\int_1^2 2rdr\right) = 4\pi\left(\left[\frac{r^3}{3}\right]_0^1+\left[\frac{r^2}{2}\right]_1^2\right) = \frac{22}{3}\pi.$$

(7)　$\displaystyle\iiint_D \frac{xyz}{(x^2+y^2+z^2)^{\frac{5}{2}}}dxdydz = \int_0^4 dz\int_0^{3\sqrt{1-\frac{z^2}{16}}}dy\int_0^{2\sqrt{1-\frac{y^2}{9}-\frac{z^2}{16}}}\frac{xyz}{(x^2+y^2+z^2)^{\frac{5}{2}}}dz$

$$= \int_0^4 dz \int_0^{3\sqrt{1-\frac{z^2}{16}}} yz \left[-\frac{1}{3(x^2+y^2+z^2)^{\frac{3}{2}}} \right]_{x=0}^{x=2\sqrt{1-\frac{y^2}{9}-\frac{z^2}{16}}} dy$$

$$= \frac{1}{3} \int_0^4 dz \int_0^{3\sqrt{1-\frac{z^2}{16}}} \left\{ \frac{1}{(y^2+z^2)^{\frac{3}{2}}} - \frac{1}{(4+\frac{5}{9}y^2+\frac{3}{4}z^2)^{\frac{3}{2}}} \right\} dy$$

$$= \frac{1}{3} \int_0^4 z \left[-\frac{1}{\sqrt{y^2+z^2}} + \frac{9}{5} \frac{1}{\sqrt{4+\frac{5}{9}y^2+\frac{3}{4}z^2}} \right]_{y=0}^{y=3\sqrt{1-\frac{x^2}{16}}} dz$$

$$= \frac{1}{3} \int_0^4 z \left(\frac{9}{5} \frac{1}{\sqrt{9+\frac{7}{16}z^2}} + \frac{1}{z} - \frac{9}{5} \frac{1}{\sqrt{4+\frac{3}{4}z^2}} - \frac{1}{\sqrt{9+\frac{7}{16}z^2}} \right) dz$$

$$= \frac{1}{3} \int_0^4 \left(\frac{16}{5} \frac{z}{\sqrt{144+7z^2}} + 1 - \frac{18}{5} \frac{z}{\sqrt{16+3z^2}} \right) dz$$

$$= \frac{1}{3} \left[\frac{16}{35}\sqrt{144+7z^2} + z - \frac{6}{5}\sqrt{16+3z^2} \right]_0^4 = \frac{12}{35}.$$

(8)　$x = u^2,\ y = v^2,\ z = w^2$ により，$E\colon u+v+w \leqq 1,\ u \geqq 0,\ v \geqq 0,\ w \geqq 0$ は $\tilde{D}\colon \sqrt{x}+\sqrt{y}+\sqrt{z} \leqq 1,\ x \geqq 0,\ y \geqq 0,\ z \geqq 0$ に写り，$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{pmatrix} = 8uvw$. 対称性より，

$$\iiint_D \left(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|} \right) dxdydz = 8 \iiint_{\tilde{D}} \left(\sqrt{x} + \sqrt{y} + \sqrt{z} \right) dz$$

$$= 64 \iiint_E (u+v+w)uvw dudvdw = 64 \int_0^1 du \int_0^{1-u} dv \int_0^{1-u-v} uv \left\{ (u+v)w + w^2 \right\} dw$$

$$= 64 \int_0^1 du \int_0^{1-u} uv \left[(u+v)\frac{w^2}{2} + \frac{w^3}{3} \right]_{w=0}^{w=1-u-v} dv$$

$$= \frac{32}{3} \int_0^1 du \int_0^{1-u} uv \left\{ 3(1-u-v)^2 - (1-u-v)^3 \right\} dv$$

$$= \frac{32}{3} \int_0^1 u \int_0^{1-u} \left\{ (1-u-v)^3 - \frac{(1-u-v)^4}{4} \right\} dvdu$$

$$= \frac{8}{3} \int_0^1 u \left[-(1-u-v)^4 + \frac{(1-u-v)^5}{5} \right]_{v=0}^{v=1-u} du = \frac{8}{15} \int_0^1 u \left\{ 5(1-u)^4 - (1-u)^5 \right\} du$$

$$= \frac{8}{15} \int_0^1 \left\{ 5(1-u)^4 - 6(1-u)^5 + (1-u)^6 \right\} du$$

$$= \frac{8}{15} \left[-(1-u)^5 + (1-u)^6 - \frac{1}{7}(1-u)^7 \right]_0^1 = \frac{8}{105}.$$

(9)　$D_n\colon x^2+y^2+z^2 \leqq n^2$ とすると，対称性より，

$$\iiint_{\mathbb{R}^3} e^{-\sqrt{x^2+y^2+z^2}} \sin \sqrt{x^2+y^2+z^2} dxdydz$$

$$= \lim_{n\to\infty} \iiint_{D_n} e^{-\sqrt{x^2+y^2+z^2}} \sin \sqrt{x^2+y^2+z^2} dxdydz$$

$$= 8 \lim_{n\to\infty} \int_0^n dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} e^{-r} (\sin r) r^2 \sin\theta d\varphi = 4\pi \lim_{n\to\infty} \left(\int_0^n e^{-r} r^2 \sin r dr \right) \left(\int_0^{\frac{\pi}{2}} \sin\theta d\theta \right)$$

$$= 4\pi \lim_{n\to\infty} \int_0^n e^{-r} r^2 \sin r dr \left[-\cos\theta\right]_0^{\frac{\pi}{2}} = 4\pi \lim_{n\to\infty} \int_0^n e^{-r} r^2 \sin r dr.$$

ここで，2. (6) と同じ記号を用いると，

$$\begin{aligned} K = \int e^{-r} r^2 \sin r dr &= -e^{-r} r^2 \sin r + \int e^{-r} \left(2r \sin r + r^2 \cos r\right) dr \\ &= -e^{-r} r^2 \sin r + 2J - e^{-r} r^2 \cos r + \int e^{-r} \left(2r \cos r - r^2 \sin r\right) dr \\ &= -e^{-r} r^2 (\sin r + \cos r) + 2J + 2\int e^{-r} r \cos r dr - K. \end{aligned}$$

また，2. (6) と同様の計算で，$\displaystyle\int e^{-r} r \cos r dr = \frac{1}{2} e^{-r} \left\{(r+1)\sin r - r\cos r\right\}$ なので，

$$K = -\frac{1}{2} e^{-r} r^2 (\sin r + \cos r) + \frac{1}{2} e^{-r} \left\{\sin r - (2r+1)\cos r\right\}.$$

よって，

$$\begin{aligned} &\iiint_{\mathbb{R}^3} e^{-\sqrt{x^2+y^2+z^2}} \sin\sqrt{x^2+y^2+z^2} dxdydz \\ &= 4\pi \lim_{n\to\infty} \left[-\frac{1}{2} e^{-r} r^2 (\sin r + \cos r) + \frac{1}{2} e^{-r} \left\{\sin r - (2r+1)\cos r\right\}\right]_0^n = 2\pi. \end{aligned}$$

5.

(1)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1-x$ であり，$y = (1-x)v$ とすると，$\dfrac{dy}{dv} = 1-x$ なので，

$$\begin{aligned} &\iint_D x^{p-1} y^{q-1} (1-x-y)^{r-1} dxdy = \int_0^1 dx \int_0^{1-x} x^{p-1} y^{q-1} (1-x-y)^{r-1} dy \\ &= \left(\int_0^1 x^{p-1} (1-x)^{q+r-1} dx\right) \left(\int_0^1 v^{q-1} (1-v)^{r-1} dv\right) = B(p, q+r) B(q, r) = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}. \end{aligned}$$

(2)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1-x,\ 0 \leqq z \leqq 1-x-y$ であり，$z = (1-x-y)w$ とすると，$\dfrac{dz}{dw} = 1-x-y$ なので，(1) より，

$$\begin{aligned} &\iiint_D x^{p-1} y^{q-1} z^{r-1} (1-x-y-z)^{s-1} dxdydz \\ &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} x^{p-1} y^{q-1} z^{r-1} (1-x-y-z)^{s-1} dz \\ &= \left(\int_0^1 dx \int_0^{1-x} x^{p-1} y^{q-1} (1-x-y)^{r+s-1} dy\right) \left(\int_0^1 w^{r-1} (1-w)^{s-1} dw\right) \\ &= \frac{\Gamma(p)\Gamma(q)\Gamma(r+s)}{\Gamma(p+q+r+s)} B(r, s) = \frac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{\Gamma(p+q+r+s)}. \end{aligned}$$

6.　求める曲面積を S とする．

(1)　対称性より，$z \geqq 0$ としてよい．$(x^2+y^2)^{\frac{1}{3}} + z^{\frac{2}{3}} = 1$ の両辺を x，y で偏微分して，

$$z_x = -\frac{xz^{\frac{1}{3}}}{(x^2+y^2)^{\frac{2}{3}}},\quad z_y = -\frac{yz^{\frac{1}{3}}}{(x^2+y^2)^{\frac{2}{3}}}$$

なので，$\sqrt{1+z_x^2+z_y^2}=(x^2+y^2)^{-\frac{1}{6}}$．$D\colon x^2+y^2\leqq 1$ として，極座標変換より，

$$S=2\iint_D (x^2+y^2)^{-\frac{1}{6}}dxdy=2\int_0^1 dr\int_0^{2\pi} r^{\frac{2}{3}}d\theta=4\pi\left[\frac{3}{5}r^{\frac{5}{3}}\right]_0^1=\frac{12}{5}\pi.$$

(2)　$x=r\cos\theta$，$y=r\sin\theta$ とすると，$(x^2+y^2)^2=x^2-y^2$ の第 1 象限にある部分は $r=\sqrt{\cos 2\theta}$ $\left(0\leqq\theta\leqq\frac{\pi}{4}\right)$ とかける．このとき，

$$\frac{dr}{d\theta}=-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}},\quad \frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta=-\frac{1}{\sqrt{\cos 2\theta}}\sin 3\theta,$$
$$\frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta=\frac{1}{\sqrt{\cos 2\theta}}\cos 3\theta,\quad 1+\left(\frac{dy}{dx}\right)^2=1+\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)^2=\frac{1}{\sin^2 3\theta}$$

なので，

$$S=2\pi\int_0^{\frac{\pi}{4}}\sin\theta d\theta=2\pi\left[-\cos\theta\right]_0^{\frac{\pi}{4}}=\left(2-\sqrt{2}\right)\pi.$$

(3)　$0\leqq x\leqq 1$ では，$y'=-\frac{1-\sqrt{x}}{\sqrt{x}}$ なので，対称性より，

$$S=4\pi\int_0^1(1-\sqrt{x})^2\sqrt{\frac{2x-2\sqrt{x}+1}{x}}dx.$$

ここで，$x=\left(t+\frac{1}{2}\right)^2$ とすると，$\frac{dx}{dt}=2t+1$ なので，

$$S=2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}(2t-1)^2\sqrt{2t^2+\frac{1}{2}}dt=2\pi\left\{4\sqrt{2}\int_0^{\frac{1}{2}}t^2\sqrt{4t^2+1}dt+\sqrt{2}\int_0^{\frac{1}{2}}\sqrt{4t^2+1}dt\right\}.$$

また，

$$\begin{aligned}I&=\int_0^{\frac{1}{2}}t^2\sqrt{4t^2+1}dt=\left[\frac{t^3}{3}\sqrt{4t^2+1}\right]_0^{\frac{1}{2}}-\frac{1}{3}\int_0^{\frac{1}{2}}t^3\frac{4t}{\sqrt{4t^2+1}}dt\\&=\frac{\sqrt{2}}{24}-\frac{1}{3}\int_0^{\frac{1}{2}}\left(t^2\sqrt{4t^2+1}-\frac{t^2}{\sqrt{4t^2+1}}\right)dt\\&=\frac{\sqrt{2}}{24}-\frac{1}{3}I+\frac{1}{12}\int_0^{\frac{1}{2}}\sqrt{4t^2+1}dt-\frac{1}{12}\int_0^{\frac{1}{2}}\frac{dt}{\sqrt{4t^2+1}}\end{aligned}$$

なので，$I=\frac{\sqrt{2}}{32}+\frac{1}{16}\int_0^{\frac{1}{2}}\sqrt{4t^2+1}dt-\frac{1}{16}\int_0^{\frac{1}{2}}\frac{dt}{\sqrt{4t^2+1}}$．よって，

$$\begin{aligned}S&=2\pi\left\{\frac{1}{4}+\frac{5\sqrt{2}}{4}\int_0^{\frac{1}{2}}\sqrt{4t^2+1}dt-\frac{\sqrt{2}}{4}\int_0^{\frac{1}{2}}\frac{dt}{\sqrt{4t^2+1}}\right\}\\&=2\pi\left\{\frac{\sqrt{5}}{8}+\frac{7\sqrt{2}}{4}\left[\frac{t\sqrt{4t^2+1}}{2}+\frac{1}{4}\log\left(2t+\sqrt{4t^2+1}\right)\right]_0^{\frac{1}{2}}\right.\\&\qquad\left.-\frac{9\sqrt{2}}{4}\left[\frac{1}{4}\log\left(2t+\sqrt{4t^2+1}\right)\right]_0^{\frac{1}{2}}\right\}\\&=\frac{\pi}{8}\left\{14+3\sqrt{2}\log\left(1+\sqrt{2}\right)\right\}.\end{aligned}$$

(4) $S = 2\pi \int_0^{\frac{\pi}{4}} \tan x \sqrt{1 + \frac{1}{\cos^4 x}} dx = 2\pi \int_0^{\frac{\pi}{4}} \frac{\sin x \sqrt{1+\cos^4 x}}{\cos^3 x} dx.$

ここで，$t = \cos x$ と変換して，$S = 2\pi \int_{\frac{1}{\sqrt{2}}}^1 \frac{\sqrt{t^4+1}}{t^3} dt$．さらに，$t = u^{\frac{1}{4}}$ とすると，$\frac{dt}{du} = \frac{1}{4} u^{-\frac{3}{4}}$ なので，

$$\begin{aligned} S &= \frac{\pi}{2} \int_{\frac{1}{4}}^1 u^{-\frac{3}{2}} \sqrt{u+1} du = \frac{\pi}{2} \left(\left[-2u^{-\frac{1}{2}} \sqrt{u+1} \right]_{\frac{1}{4}}^1 + \int_{\frac{1}{4}}^1 \frac{du}{\sqrt{u^2+u}} \right) \\ &= \pi \left(\sqrt{5} - \sqrt{2} \right) + \frac{\pi}{2} \left[2 \log \left(\sqrt{x} + \sqrt{x+1} \right) \right]_{\frac{1}{4}}^1 = \pi \left\{ \sqrt{5} - \sqrt{2} + \log \frac{2\left(1+\sqrt{2}\right)}{1+\sqrt{5}} \right\}. \end{aligned}$$

(5) 対称性より，$z \geqq 0$ としてよい．

$$z_x = x \frac{1 - 2\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2} - x^2 - y^2}}, \quad z_y = y \frac{1 - 2\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2} - x^2 - y^2}}$$

なので，$\sqrt{1 + z_x^2 + z_y^2} = \frac{1}{2\sqrt{\sqrt{x^2+y^2} - x^2 y^2}}$．$D\colon x^2 + y^2 \leqq 1$ とし，極座標変換より，

$$\begin{aligned} S &= \iint_D \frac{dxdy}{\sqrt{\sqrt{x^2+y^2} - x^2 - y^2}} = \int_0^1 dr \int_0^{2\pi} \frac{r}{\sqrt{r - r^2}} d\theta \\ &= 2\pi \int_0^1 r^{\frac{1}{2}} (1-r)^{-\frac{1}{2}} dr = 2\pi B\left(\frac{3}{2}, \frac{1}{2} \right) = 2\pi \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \pi^2. \end{aligned}$$

(6) $x = r\cos\theta$，$y = r\sin\theta$ とすると，$y > 0$ の部分は $r = 1 + \cos\theta$ $(0 \leqq \theta \leqq \pi)$ とかけるので，$x = \cos\theta + \cos^2\theta$，$y = \sin\theta + \sin\theta\cos\theta$．従って，$\frac{dx}{d\theta} = -\sin\theta - \sin 2\theta$，$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$ であり，

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \right)^2 = \frac{4\cos^2 \frac{\theta}{2}}{(\sin\theta + \sin 2\theta)^2}$$

なので，

$$S = 2\pi \int_0^\pi \sin\theta (1 + \cos\theta) 2\cos\frac{\theta}{2} d\theta = 8\pi \int_0^\pi \sin\theta \cos^3 \frac{\theta}{2} d\theta.$$

ここで，$\theta = 2\varphi$ として，

$$S = 16\pi \int_0^{\frac{\pi}{2}} \sin 2\varphi \cos^3 \varphi d\varphi = 32\pi \int_0^{\frac{\pi}{2}} \sin\varphi \cos^4 \varphi d\varphi = 16\pi B\left(1, \frac{5}{2}\right) = 16\pi \frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{7}{2})} = \frac{32}{5}\pi.$$

(7) $z_x = -\left(\frac{z}{x} \right)^{\frac{1}{3}}$，$z_y = -\left(\frac{z}{y} \right)^{\frac{1}{3}}$ なので，$1 + z_x^2 + z_y^2 = x^{-\frac{2}{3}} + y^{-\frac{2}{3}} - x^{-\frac{2}{3}} y^{\frac{2}{3}} - x^{\frac{2}{3}} y^{-\frac{2}{3}} - 1$．$x = \tilde{x}^{\frac{3}{2}}$，$y = \tilde{y}^{\frac{3}{2}}$ により，$E\colon \tilde{x} + \tilde{y} \leqq 1, \tilde{x} \geqq 0, \tilde{y} \geqq 0$ は $D\colon x^{\frac{2}{3}} + y^{\frac{2}{3}}, x \geqq 0, y \geqq 0$ に写り，$\frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})} = \det \begin{pmatrix} \frac{3}{2}\tilde{x}^{\frac{1}{2}} & 0 \\ 0 & \frac{3}{2}\tilde{y}^{\frac{1}{2}} \end{pmatrix} = \frac{9}{4}\sqrt{\tilde{x}\tilde{y}}$．対称性より，

$$S = 8 \iint_D \sqrt{1 + z_x^2 + z_y^2} dxdy = 18 \int_E \sqrt{\tilde{x} + \tilde{y} - (\tilde{x}+\tilde{y})^2 + \tilde{x}\tilde{y}} d\tilde{x} d\tilde{y}.$$

対称性より，$\tilde{x} \geqq \tilde{y}$ の場合を考えればよい．$u=\tilde{x}+\tilde{y}$，$v=\tilde{x}\tilde{y}$ とすると，$\dfrac{\partial(u,v)}{\partial(\tilde{x},\tilde{y})}=\det\begin{pmatrix}1 & 1\\ \tilde{y} & \tilde{x}\end{pmatrix}=\tilde{x}-\tilde{y}=\sqrt{u^2-4v}$ なので，$S=36\displaystyle\int_0^1 du\int_0^{\frac{u^2}{4}}\sqrt{\frac{u-u^2+v}{u^2-4v}}dv$．ここで，$t=\sqrt{\dfrac{u-u^2+v}{u^2-4v}}$ とすると，$v=\dfrac{u^2t^2+u^2-u}{4t^2+1}$ より，$\dfrac{dv}{dt}=\dfrac{2(-3u+4)ut}{(4t^2+1)^2}$ なので，

$$\begin{aligned}
\int_0^{\frac{u^2}{4}}\sqrt{\frac{u-u^2+v}{u^2-4v}}dv &= 2(-3u+4)u\int_{\sqrt{\frac{1-u}{u}}}^{\infty}\frac{t^2}{(4t^2+1)^2}dt\\
&= 2(-3u+4)u\lim_{n\to\infty}\left(\left[-\frac{t}{8(4t^2+1)}\right]_{\sqrt{\frac{1-u}{u}}}^{n}+\frac{1}{8}\int_{\sqrt{\frac{1-u}{u}}}^{n}\frac{dt}{4t^2+1}\right)\\
&= \frac{u\sqrt{u-u^2}}{4}+\frac{u(-3u+4)}{4}\lim_{n\to\infty}\left[\frac{1}{2}\tan^{-1}2t\right]_{\sqrt{\frac{1-u}{u}}}^{n}\\
&= \frac{1}{4}u^{\frac{3}{2}}(1-u)^{\frac{1}{2}}+\frac{\pi(-3u^2+4u)}{16}-\frac{-3u^2+4u}{8}\tan^{-1}2\sqrt{\frac{1-u}{u}}.
\end{aligned}$$

ゆえに，

$$\begin{aligned}
S &= 9B\left(\frac{5}{2},\frac{3}{2}\right)+\frac{9}{4}\pi\int_0^1(-3u^2+4u)du-\frac{9}{2}\int_0^1(4u-3u^2)\tan^{-1}2\sqrt{\frac{1-u}{u}}du\\
&= 9\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}+\frac{9}{4}\pi\left[-u^3+2u^2\right]_0^1-\frac{9}{2}\int_0^1(4u-3u^2)\tan^{-1}2\sqrt{\frac{1-u}{u}}du\\
&= \frac{45}{16}\pi-\frac{9}{2}\int_0^1(4u-3u^2)\tan^{-1}2\sqrt{\frac{1-u}{u}}du.
\end{aligned}$$

また，

$$\begin{aligned}
I &= \int_0^1(4u-3u^2)\tan^{-1}2\sqrt{\frac{1-u}{u}}du\\
&= \left[(2u^2-u^3)\tan^{-1}2\sqrt{\frac{1-u}{u}}\right]_0^1+\int_0^1\frac{2u-u^2}{4-3u}\sqrt{\frac{u}{1-u}}du=\int_0^1\frac{2u-u^2}{4-3u}\sqrt{\frac{u}{1-u}}du
\end{aligned}$$

であり，$w=\sqrt{\dfrac{u}{1-u}}$ とすると，$u=\dfrac{w^2}{w^2+1}$，$\dfrac{du}{dw}=\dfrac{2w}{(w^2+1)^2}$ より，

$$\begin{aligned}
I &= 2\int_0^{\infty}\frac{(w^2+2)w^4}{(w^2+4)(w^2+1)^3}dw\\
&= 2\int_0^{\infty}\left\{\frac{32}{27(w^2+4)}-\frac{5}{27(w^2+1)}-\frac{4}{9(w^2+1)^2}+\frac{1}{3(w^2+1)^3}\right\}dw.
\end{aligned}$$

ここで，

$$\begin{aligned}
\int\frac{dw}{(w^2+1)^2} &= \int\frac{dw}{w^2+1}-\int\frac{w}{2}\frac{2w}{(w^2+1)^2}dw=\int\frac{dw}{w^2+1}+\frac{w}{2}\frac{1}{w^2+1}-\frac{1}{2}\int\frac{dw}{w^2+1}\\
&= \frac{w}{2(w^2+1)}+\frac{1}{2}\int\frac{dw}{w^2+1},
\end{aligned}$$

$$\begin{aligned}
\int\frac{dw}{(w^2+1)^3} &= \int\frac{dw}{(w^2+1)^2}-\int\frac{w}{4}\frac{4w}{(w^2+1)^3}dw\\
&= \int\frac{dw}{(w^2+1)^2}+\frac{w}{4(w^2+1)^2}-\frac{1}{4}\int\frac{dw}{(w^2+1)^2}=\frac{w}{4(w^2+1)^2}+\frac{3}{4}\int\frac{dw}{(w^2+1)^2}\\
&= \frac{w}{4(w^2+1)^2}+\frac{3}{4}\frac{w}{2(w^2+1)}+\frac{3}{8}\int\frac{dw}{w^2+1}
\end{aligned}$$

より，

$$I = 2\lim_{n\to\infty}\left(\left[\frac{1}{3}\frac{w}{4(w^2+1)^2} - \frac{7}{36}\frac{w}{2(w^2+1)}\right]_0^n + \int_0^n \left\{\frac{32}{27(w^2+4)} - \frac{61}{216}\frac{1}{w^2+1}\right\} dw\right)$$
$$= 2\lim_{n\to\infty}\left[\frac{16}{27}\tan^{-1}\frac{w}{2} - \frac{61}{216}\tan^{-1} w\right]_0^n = \frac{67}{216}\pi$$

なので，$S = \frac{45}{16}\pi - \frac{9}{2}I = \frac{45}{16}\pi - \frac{67}{48}\pi = \frac{17}{12}\pi.$

7.　全質量を M とし，求める重心を $G(x_0, y_0, z_0)$，慣性モーメントを I とする．

(1)　対称性より，$G(0,0,0)$，

$$I = \iiint_D |xyz|(x^2+z^2)dxdydz = 8\int_0^1 dx\int_0^2 dy\int_0^3 xy(x^2z+z^3)dz$$
$$= 8\int_0^1 x\left[\frac{x^2}{2}z^2 + \frac{z^4}{4}\right]_{z=0}^{z=3} dx\left[\frac{y^2}{2}\right]_0^2 = 16\int_0^1\left(\frac{9}{2}x^3 + \frac{81}{4}x\right)dx = 16\left[\frac{9}{8}x^4 + \frac{81}{8}x^2\right]_0^1 = 180.$$

(2)　$x = r\cos\theta$，$y = r\sin\theta$ とすると，

$$M = \iiint_D \sqrt{x^2+y^2}dxdydz = \left(\int_0^1 dz\right)\left(\int_0^1 dr\int_0^{2\pi} r^2 d\theta\right) = 2\pi\left[\frac{r^3}{3}\right]_0^1 = \frac{2}{3}\pi.$$

対称性より，$x_0 = y_0 = 0$.

$$z_0 = \frac{1}{M}\iiint_D \sqrt{x^2+y^2}zdxdydz = \frac{3}{2\pi}\left(\int_0^1 zdz\right)\left(\int_0^1 dr\int_0^{2\pi} r^2 d\theta\right) = 3\left[\frac{z^2}{2}\right]_0^1\left[\frac{r^3}{3}\right]_0^1 = \frac{1}{2}.$$

よって，$G\left(0, 0, \frac{1}{2}\right)$.

$$I = \iiint_D (x^2+y^2)^{\frac{3}{2}}dxdydz = \left(\int_0^1 dz\right)\left(\int_0^1 dr\int_0^{2\pi} r^4 d\theta\right) = 2\pi\left[\frac{z^2}{2}\right]_0^1\left[\frac{r^5}{5}\right]_0^1 = \frac{2}{5}\pi.$$

(3)　$M = \iiint_D x^2z^2dxdydz = \int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} x^2z^2dz$

$$= \frac{1}{3}\int_0^1 dx\int_0^{1-x} x^2(1-x-y)^3dy = \frac{1}{3}\int_0^1 x^2\left[\frac{(1-x-y)^4}{4}\right]_{y=0}^{y=1-x} dx$$
$$= \frac{1}{12}\int_0^1 x^2(1-x)^4dx = \frac{1}{12}B(3,5) = \frac{1}{12}\frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{1}{1260}.$$

同様の計算で，

$$x_0 = \frac{1}{M}\iiint_D x^3z^2dxdydz = 105\int_0^1 x^3(1-x)^4dx = 105B(4,5) = 105\frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3}{8},$$
$$y_0 = \frac{1}{M}\iiint_D x^2yz^2dxdydz = 420\int_0^1 dx\int_0^{1-x} x^2y(1-x-y)^3dy$$
$$= 420\int_0^1 x^2\left\{\left[-y\frac{(1-x-y)^4}{4}\right]_{y=0}^{y=1-x} + \frac{1}{4}\int_0^{1-x}(1-x-y)^4dy\right\}dx$$
$$= 105\int_0^1 x^2\left[-\frac{(1-x-y)^5}{5}\right]_{y=0}^{y=1-x} dx = 21\int_0^1 x^2(1-x)^5dx$$
$$= 21B(3,6) = 21\frac{\Gamma(3)\Gamma(6)}{\Gamma(9)} = \frac{1}{8},$$

$$
\begin{aligned}
z_0 &= \frac{1}{M}\iiint_D x^2z^3dxdydz = 1260\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}x^2z^3dz \\
&= 1260\int_0^1 dx\int_0^{1-x}x^2\left[\frac{z^4}{4}\right]_0^{1-x-y}dy = 315\int_0^1 dx\int_0^{1-x}x^2(1-x-y)^4dy \\
&= 315\int_0^1 x^2\left[-\frac{(1-x-y)^5}{5}\right]_{y=0}^{y=1-x}dx = 63\int_0^1 x^2(1-x)^5dx \\
&= 63B(3,6) = 63\frac{\Gamma(3)\Gamma(6)}{\Gamma(9)} = \frac{3}{8}.
\end{aligned}
$$

よって，$G\left(\frac{3}{8},\frac{1}{8},\frac{3}{8}\right)$.

$$
\begin{aligned}
I &= \iiint_D (x^2+z^2)x^2z^2dxdydz = \int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}x^2(x^2z^2+z^4)dz \\
&= \int_0^1 dx\int_0^{1-x}x^2\left[x^2\frac{z^3}{3}+\frac{z^5}{5}\right]_{z=0}^{z=1-x-y}dy \\
&= \int_0^1 dx\int_0^{1-x}x^2\left\{\frac{y^2}{3}(1-x-y)^3+\frac{(1-x-y)^5}{5}\right\}dy.
\end{aligned}
$$

ここで，$y=(1-x)v$ とすると，$\frac{dy}{dv}=1-x$ なので，

$$
\begin{aligned}
I &= \int_0^1 dx\int_0^1 x^2(1-x)^6\left\{\frac{1}{3}v^2(1-v)^3+\frac{1}{5}v^5\right\}dv \\
&= B(3,7)\left\{\frac{1}{3}B(3,4)+\frac{1}{30}\right\} = \frac{\Gamma(3)\Gamma(7)}{\Gamma(10)}\left\{\frac{1}{3}\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}+\frac{1}{30}\right\} = \frac{1}{6480}.
\end{aligned}
$$

(4)　極座標変換より，

$$
\begin{aligned}
M &= \iiint_D yzdxdydz = \int_0^1 dr\int_0^{\frac{\pi}{2}}d\theta\int_0^{\pi}r^4\sin^2\theta\cos\theta\sin\varphi d\varphi \\
&= \left(\int_0^1 r^4dr\right)\left(\int_0^{\frac{\pi}{2}}\sin^2\theta\cos\theta d\theta\right)\left(\int_0^{\pi}\sin\varphi d\varphi\right) = \frac{1}{2}\left[\frac{r^5}{5}\right]_0^1 B\left(\frac{3}{2},1\right)[-\cos\varphi]_0^{\pi} \\
&= \frac{1}{5}\frac{\Gamma(\frac{3}{2})\Gamma(1)}{\Gamma(\frac{5}{2})} = \frac{2}{15}.
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= \frac{1}{M}\iiint_D y^2zdxdydz = \frac{15}{2}\left(\int_0^1 r^5dr\right)\left(\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta d\theta\right)\left(\int_0^{\pi}\frac{1-\cos2\varphi}{2}d\varphi\right) \\
&= \frac{15}{8}\left[\frac{r^6}{6}\right]_0^1 B(2,1)\left[\varphi-\frac{\sin2\varphi}{2}\right]_0^{\pi} = \frac{5}{32}\pi.
\end{aligned}
$$

対称性より，$x_0=0$，$z_0=\frac{5}{32}\pi$．よって，$G\left(0,\frac{5}{32}\pi,\frac{5}{32}\pi\right)$.

$$
\begin{aligned}
I &= \iiint_D yz(y^2+z^2)dxdydz \\
&= \left(\int_0^1 r^6dr\right)\left(\int_0^{\frac{\pi}{2}}d\theta\int_0^{\pi}\sin^2\theta\cos\theta\sin\varphi(\sin^2\theta\sin^2\varphi+\cos^2\theta)d\varphi\right) \\
&= \frac{1}{7}\int_0^{\frac{\pi}{2}}\sin^2\theta\cos\theta\int_0^{\pi}\left(\sin^2\theta\frac{3\sin\varphi-\sin3\varphi}{4}+\cos^2\theta\sin\varphi\right)d\varphi d\theta
\end{aligned}
$$

$$= \frac{1}{7}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta\left(\frac{\sin^2\theta}{4}\left[-3\cos\varphi + \frac{\cos 3\varphi}{3}\right]_0^{\pi} + \cos^2\theta\left[-\cos\varphi\right]_0^{\pi}\right)d\theta$$
$$= \frac{4}{21}\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta d\theta + \frac{2}{7}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta d\theta = \frac{2}{21}B\left(\frac{5}{2},1\right) + \frac{1}{7}B\left(\frac{3}{2},2\right) = \frac{8}{105}.$$

(5)　$x = 2r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = 3r\cos\theta$ と変換すると，

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \det\begin{pmatrix} 2\sin\theta\cos\varphi & 2r\cos\theta\cos\varphi & -2r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ 3\cos\theta & -3r\sin\theta & 0 \end{pmatrix} = 6r^2\sin\theta$$

より，

$$M = \iiint_D (x^2+y^2)dxdydz = 6\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} r^4\sin^3\theta(1+3\cos^2\varphi)d\theta$$
$$= 6\left(\int_0^1 r^4 dr\right)\left(\int_0^{\pi}\frac{3\sin\theta - \sin 3\theta}{4}d\theta\right)\left(\int_0^{\frac{\pi}{2}}\frac{3\cos 2\varphi + 5}{2}d\varphi\right)$$
$$= \frac{3}{4}\left[\frac{r^5}{5}\right]_0^1\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\frac{\pi}{2}}\left[\frac{3}{2}\sin 2\varphi + 5\varphi\right]_0^{\frac{\pi}{2}} = \pi.$$

$$x_0 = \frac{1}{M}\iiint_D x(x^2+y^2)dxdydz$$
$$= \frac{12}{\pi}\left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}}(1+3\cos^2\varphi)\cos\varphi d\varphi\right)$$
$$= \frac{6}{\pi}\left[\frac{r^6}{6}\right]_0^1 B\left(\frac{5}{2},\frac{1}{2}\right)\left\{\left[\sin\varphi\right]_0^{\frac{\pi}{2}} + \frac{3}{2}B\left(\frac{1}{2},2\right)\right\} = \frac{1}{\pi}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}\left\{1 + \frac{3}{2}\frac{\Gamma(\frac{1}{2})\Gamma(2)}{\Gamma(\frac{5}{2})}\right\} = \frac{9}{8},$$

$$y_0 = \frac{1}{M}\iiint_D y(x^2+y^2)dxdydz = \frac{6}{\pi}\left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}}(1+3\cos^2\varphi)\sin\varphi d\varphi\right)$$
$$= \frac{3}{\pi}\left[\frac{r^6}{6}\right]_0^1 B\left(\frac{5}{2},\frac{1}{2}\right)\left\{\left[-\cos\varphi\right]_0^{\frac{\pi}{2}} + \frac{3}{2}B\left(1,\frac{3}{2}\right)\right\}$$
$$= \frac{1}{2\pi}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}\left\{1 + \frac{3}{2}\frac{\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\right\} = \frac{3}{8},$$

$$z_0 = \frac{1}{M}\iiint_D z(x^2+y^2)dxdydz = \frac{18}{\pi}\left(\int_0^1 r^5 dr\right)\left(\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}}(1+3\cos^2\varphi)d\varphi\right)$$
$$= \frac{9}{2\pi}\left[\frac{r^6}{6}\right]_0^1 B(2,1)\left[5\varphi + \frac{3\sin 2\varphi}{2}\right]_0^{\frac{\pi}{2}} = \frac{15}{16}.$$

よって，$G\left(\frac{9}{8},\frac{3}{8},\frac{15}{16}\right)$.

$$I = \iiint_D (y^2+z^2)(x^2+y^2)dxdydz$$
$$= 6\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} r^6\sin^3\theta((9-\sin^2\varphi)\cos^2\theta + \sin^2\varphi)(1+3\cos^2\varphi)d\varphi$$
$$= 6\left[\frac{r^7}{7}\right]_0^1\int_0^{\frac{\pi}{2}}\left\{(9-\sin^2\varphi)\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta d\theta + \sin^2\varphi\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta\right\}(1+3\cos^2\varphi)d\varphi$$
$$= \frac{6}{7}\int_0^{\frac{\pi}{2}}\left\{(9-\sin^2\varphi)\frac{1}{2}B\left(2,\frac{3}{2}\right) + \sin^2\varphi\frac{1}{2}B\left(2,\frac{1}{2}\right)\right\}(1+3\cos^2\varphi)d\varphi$$

$$= \frac{4}{35}\int_0^{\frac{\pi}{2}}(13+23\cos^2\varphi+12\sin^2\varphi\cos^2\varphi)d\varphi = \frac{4}{35}\left\{\frac{13}{2}\pi+\frac{23}{2}B\left(\frac{3}{2},\frac{1}{2}\right)+6B\left(\frac{3}{2},\frac{3}{2}\right)\right\}$$
$$= \frac{4}{35}\left\{\frac{13}{2}\pi+\frac{23}{2}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}+6\frac{\Gamma(\frac{3}{2})^2}{\Gamma(3)}\right\} = \frac{52}{35}\pi.$$

(6) $x = u^2$, $y = v^2$, $z = w^2$ とすると，$E\colon u+v+w \leqq 1,\ u \geqq 0,\ v \geqq 0,\ w \geqq 0$ は D に写り，
$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{pmatrix} = 8uvw$ なので，

$$M = \iiint_D 2dxdydz = 16\int_0^1 du\int_0^{1-u} dv\int_0^{1-u-v} uvwdw$$
$$= 16\int_0^1 du\int_0^{1-u} uv\left[\frac{w^2}{2}\right]_0^{1-u-v} dv = 8\int_0^1 du\int_0^{1-u} uv(1-u-v)^2dv$$
$$= 8\int_0^1 u\left\{\left[-v\frac{(1-u-v)^3}{3}\right]_{v=0}^{v=1-u}+\frac{1}{3}\int_0^{1-u}(1-u-v)^3dv\right\}$$
$$= \frac{8}{3}\int_0^1 u\left[-\frac{(1-u-v)^4}{4}\right]_{v=0}^{v=1-u} du = \frac{2}{3}\int_0^1 u(1-u)^4du = \frac{2}{3}B(2,5)$$
$$= \frac{2}{3}\frac{\Gamma(2)\Gamma(5)}{\Gamma(7)} = \frac{1}{45}.$$

同様の計算により，

$$x_0 = \frac{1}{M}\iiint_D 2xdxdydz = 720\int_0^1 du\int_0^{1-u} dv\int_0^{1-u-v} u^3vwdw$$
$$= 30\int_0^1 u^3(1-u)^4du = 30B(4,5) = 30\frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3}{28}.$$

対称性より，$y_0 = z_0 = \dfrac{3}{28}$. よって，$G\left(\dfrac{3}{28},\dfrac{3}{28},\dfrac{3}{28}\right)$.

$$I = \iiint_D 2(x^2+z^2)dxdydz = 16\int_0^1 du\int_0^{1-u} dv\int_0^{1-u-v}(u^4+w^4)uvwdw$$
$$= 16\int_0^1 du\int_0^{1-u}\left[u^5v\frac{w^2}{2}+uv\frac{w^6}{6}\right]_{w=0}^{w=1-u-v} dv$$
$$= \frac{8}{3}\int_0^1 du\int_0^{1-u}\left\{3u^5v(1-u-v)^2+uv(1-u-v)^6\right\}dv$$
$$= \frac{8}{3}\int_0^1 du\int_0^{1-u}\left\{u^5(1-u-v)^3+u\frac{(1-u-v)^7}{7}\right\}dv$$
$$= \frac{8}{21}\int_0^1\left[-\frac{7}{4}u^5(1-u-v)^4-u\frac{(1-u-v)^8}{8}\right]_{v=0}^{v=1-u} du$$
$$= \frac{1}{21}\int_0^1\left\{14u^5(1-u)^4+u(1-u)^8\right\}du = \frac{1}{21}\left\{14B(6,5)+B(2,9)\right\}$$
$$= \frac{1}{21}\left\{14\frac{\Gamma(6)\Gamma(5)}{\Gamma(11)}+\frac{\Gamma(2)\Gamma(9)}{\Gamma(11)}\right\} = \frac{1}{945}.$$

(7) $x = r\sin^3\theta\cos^3\varphi$, $y = r\sin^3\theta\sin^3\varphi$, $z = r\cos^3\theta$ とすると，

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \det\begin{pmatrix} \sin^3\theta\cos^3\varphi & 3r\sin^2\theta\cos\theta\cos^3\varphi & -3r\sin^3\theta\cos^2\varphi\sin\varphi \\ \sin^3\theta\sin^3\varphi & 3r\sin^2\theta\cos\varphi\sin^3\varphi & 3r\sin^3\theta\sin^2\varphi\cos\varphi \\ \cos^3\varphi & -3r\cos^2\theta\sin\theta & 0 \end{pmatrix}$$

$$= 9r^2 \sin^5\theta \cos^2\theta \sin^2\varphi \cos^2\varphi$$

なので，

$$\begin{aligned}
M &= \iiint_D dxdydz = 9\left(\int_0^1 r^2 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^2\varphi d\varphi\right) \\
&= \frac{9}{4}\left[\frac{r^3}{3}\right]_0^1 B\left(3, \frac{3}{2}\right) B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{3}{4}\frac{\Gamma(3)\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{2})}\frac{\Gamma(\frac{3}{2})^2}{\Gamma(3)} = \frac{\pi}{70}. \\
x_0 &= \frac{1}{M}\iiint_D xdxdydz = \frac{630}{\pi}\left(\int_0^1 r^3 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^8\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^5\varphi d\varphi\right) \\
&= \frac{315}{2\pi}\left[\frac{r^4}{4}\right]_0^1 B\left(\frac{9}{2}, \frac{3}{2}\right) B\left(\frac{3}{2}, 3\right) = \frac{315}{8\pi}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{3}{2})}{\Gamma(6)}\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(\frac{9}{2})} = \frac{21}{128}.
\end{aligned}$$

対称性より，$y_0 = z_0 = \dfrac{21}{128}$．よって，$G\left(\dfrac{21}{128}, \dfrac{21}{128}, \dfrac{21}{128}\right)$.

$$\begin{aligned}
I &= \iiint_D (x^2 + y^2) dxdydz \\
&= 9\left(\int_0^1 r^4 dr\right)\left(\int_0^{\frac{\pi}{2}} \sin^{11}\theta\cos^2\theta d\theta\right)\left(\int_0^{\frac{\pi}{2}} (\sin^8\varphi\cos^2\varphi + \sin^2\varphi\cos^8\varphi) d\varphi\right) \\
&= \frac{9}{4}\left[\frac{r^5}{5}\right]_0^1 B\left(6, \frac{3}{2}\right)\left\{B\left(\frac{9}{2}, \frac{3}{2}\right) + B\left(\frac{3}{2}, \frac{9}{2}\right)\right\} = \frac{9}{10}\frac{\Gamma(6)\Gamma(\frac{3}{2})}{\Gamma(\frac{15}{2})}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{3}{2})}{\Gamma(6)} = \frac{\pi}{715}.
\end{aligned}$$