5.5　重積分の応用

問 1

(1) 合成関数の偏微分より，$z_r = \cos\theta z_x + \sin\theta z_y$，$z_\theta = -r\sin\theta z_x + r\cos\theta z_y$ なので，$z_x = \cos\theta z_r - \dfrac{\sin\theta}{r}z_\theta$，$z_y = \sin\theta z_r + \dfrac{\cos\theta}{r}z_\theta$．したがって，$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+z_r^2+\dfrac{1}{r^2}z_\theta^2}$ であり，極座標変換より，

$$S = \iint_D \sqrt{1+z_x^2+z_y^2}dxdy = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} r\sqrt{1+z_r^2+\frac{1}{r^2}z_\theta^2}dr$$

が成り立つ．

(2) $x'(t) \neq 0$ なので，$x = x(t)$ の逆関数が存在する．したがって，適当な関数 $f(x)$ を用いて，$y = f(x)$ とかける．このとき，$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ なので，定理 9 で置換積分 $x = x(t)$ を行えば，$S = 2\pi\displaystyle\int_\alpha^\beta |y(t)|\sqrt{x'(t)^2+y'(t)^2}dt$ が得られる．

問 2　求める曲面積を S とする．

(1)　$z = \sqrt{1-x^2-y^2}$，$D\colon x^2+y^2 \leqq 1$ とすると，$z_x = \dfrac{x}{\sqrt{1-x^2-y^2}}$，$z_y = \dfrac{y}{\sqrt{1-x^2-y^2}}$ なので，対称性と極座標変換より，

$$S = 2\iint_D \frac{dxdy}{\sqrt{1-x^2-y^2}} = 2\int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}}dr = 4\pi\left[-\sqrt{1-r^2}\right]_0^1 = 4\pi.$$

(2)　$D\colon x^2+y^2 \leqq 1$ とすると，$z_x = 2x$，$z_y = 2y$ なので，極座標変換より，

$$\begin{aligned} S &= \iint_D \sqrt{1+4x^2+4y^2}dxdy = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+4r^2}rdr \\ &= 2\pi\left[\frac{1}{12}\left(1+4r^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right). \end{aligned}$$

(3)　$z = \sqrt{4-x^2}$，$D\colon x^2+y^2 \leqq 4$ とすると，$z_x = -\dfrac{x}{\sqrt{4-x^2}}$，$z_y = 0$ なので，対称性より，

$$S = 2\iint_D \frac{2}{\sqrt{4-x^2}}dxdy = 16\int_0^2 dx \int_0^{\sqrt{4-x^2}} \frac{dy}{\sqrt{4-x^2}} = 16\int_0^2 dx = 32.$$

(4)　$z = \sqrt{5x-x^2}$ とすると，$z_x = \dfrac{5-2x}{2\sqrt{5x-x^2}}$，$z_y = 0$ なので，対称性より，

$$S = 4\int_0^5 dx \int_0^{\sqrt{25-5x}} \frac{5}{2\sqrt{5x-x^2}}dy = 10\sqrt{5}\int_0^5 \frac{dx}{\sqrt{x}} = 10\sqrt{5}\left[2\sqrt{x}\right]_0^5 = 100.$$

(5)　球面 $x^2+y^2+z^2 = 1$ と $3x^2+4y^2 = z+1$ の交線 $x^2+y^2+(3x^2+4y^2-1)^2 = 1$ は，極座標 $x = r\cos\theta$，$y = r\sin\theta$ を用いると，$r^2+(3r^2+r^2\sin^2\theta-1)^2 = 1$．$r$ について解くと，$r = \dfrac{\sqrt{5+2\sin^2\theta}}{3+\sin^2\theta}$．ここで，$z = \sqrt{1-r^2}$ であり，$z_r = -\dfrac{r}{\sqrt{1-r^2}}$，$z_\theta = 0$ なので，$\gamma(\theta) = \dfrac{\sqrt{5+2\sin^2\theta}}{3+\sin^2\theta}$ とおけば，

$$S = \int_0^{2\pi} d\theta \int_0^{\gamma(\theta)} \frac{r}{\sqrt{1-r^2}}dr = \int_0^{2\pi} \left[-\sqrt{1-r^2}\right]_0^{\gamma(\theta)} d\theta = \int_0^{2\pi} \left(1-\frac{2+\sin^2\theta}{3+\sin^2\theta}\right)d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{d\theta}{3+\sin^2\theta} = 4\int_0^{\frac{\pi}{2}} \frac{1}{4\tan^2\theta+3}\frac{d\theta}{\cos^2\theta}.$$

ここで，$t=\tan\theta$ とすると，$\dfrac{dt}{d\theta} = \dfrac{1}{\cos^2\theta}$ なので，

$$S = \int_0^{\infty} \frac{dt}{t^2+\frac{3}{4}} = \lim_{n\to\infty} \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2}{\sqrt{3}}t\right]_0^n = \frac{2}{\sqrt{3}}\lim_{n\to\infty}\tan^{-1}\frac{2}{\sqrt{3}}n = \frac{\pi}{\sqrt{3}}.$$

(6)　$y = \pm\sqrt{4-x^2}+5$ なので，$y' = \mp\dfrac{x}{\sqrt{4-x^2}}$．したがって，

$$S = 2\pi\int_{-2}^{2}\left(\sqrt{4-x^2}+5-\sqrt{4-x^2}+5\right)\frac{2}{\sqrt{4-x^2}}dx = 40\pi\int_{-2}^{2}\frac{dx}{\sqrt{4-x^2}}$$
$$= 40\pi\left[\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 40\pi^2.$$

(7)　$y' = -\sin x$ より，

$$S = 2\pi\int_0^{\pi} |\cos x|\sqrt{1+\sin^2 x}dx = 4\pi\int_0^{\frac{\pi}{2}}\cos x\sqrt{1+\sin^2 x}dx.$$

$t = \sin x$ と変換して，

$$S = 4\pi\int_0^1 \sqrt{t^2+1}dt = 4\pi\left[\frac{1}{2}\left\{t\sqrt{t^2+1}+\log\left(t+\sqrt{t^2+1}\right)\right\}\right]_0^1 = 2\pi\left\{\sqrt{2}+\log\left(1+\sqrt{2}\right)\right\}.$$

(8)　$y = \pm\sqrt{1-\frac{x^2}{4}}$ なので，$y' = \mp\dfrac{x}{2\sqrt{4-x^2}}$ より，

$$S = \frac{\pi}{2}\int_{-2}^{2}\sqrt{16-3x^2}dx = \sqrt{3}\pi\int_0^2\sqrt{\frac{16}{3}-x^2}dx$$
$$= \sqrt{3}\pi\left[\frac{1}{2}\left(x\sqrt{\frac{16}{3}-x^2}+\frac{16}{3}\sin^{-1}\frac{\sqrt{3}}{4}x\right)\right]_0^2 = 2\pi\left(1+\frac{4}{3\sqrt{3}}\pi\right).$$

(9)　$y = \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ であり，$y' = -x^{-\frac{1}{3}}\sqrt{1-x^{\frac{2}{3}}}$ なので，

$$S = 2\pi\int_{-1}^{1}\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}x^{-\frac{1}{3}}dx = 4\pi\int_0^1\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}x^{-\frac{1}{3}}dx.$$

$x = \sin^3\theta$ とすると，$\dfrac{dx}{d\theta} = 3\sin^2\theta\cos\theta$ なので，

$$S = 12\pi\int_0^{\frac{\pi}{2}}\sin\theta\cos^4\theta d\theta = 12\pi\left[-\frac{1}{5}\cos^5\theta\right]_0^{\frac{\pi}{2}} = \frac{12}{5}\pi.$$

問 3　物体 D の全質量を M，重心を $G(x_0, y_0, z_0)$，z 軸に関する慣性モーメントを I_z とする．
(1)　$M = \iiint_D \rho dxdydz = 48\rho$．対称性より，$G(0,0,0)$．対称性より，

$$I_z = \iiint_D (x^2+y^2)\rho dxdydz = 8\rho\int_0^1 dx\int_0^2 dy\int_0^3 (x^2+y^2)dz$$

$$= 24\rho\int_0^1 dx\int_0^2 (x^2+y^2)dy = 24\rho\int_0^1 \left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=2} dx$$

$$= 24\rho\int_0^1 \left(2x^2+\frac{8}{3}\right)dx = 24\rho\left[\frac{2}{3}x^3+\frac{8}{3}\right]_0^1 = 80\rho.$$

(2) $M = \iiint_D \rho dxdydz = \rho\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} dz$

$$= \rho\int_0^1 dx\int_0^{1-x}(1-x-y)dy = \rho\int_0^1 \left[(1-x)y-\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{2}\rho\int_0^1 (1-x)^2dx = \frac{1}{2}\rho\left[-\frac{(1-x)^3}{3}\right]_0^1 = \frac{\rho}{6}.$$

$$x_0 = \frac{1}{M}\iiint_D x\rho dxdydz = 6\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} xdz$$

$$= 6\int_0^1 dx\int_0^{1-x} x(1-x-y)dy = 6\int_0^1 \left[(x-x^2)y-\frac{x}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= 3\int_0^1 (x-2x^2-x^3)dx = 3\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1 = \frac{1}{4}.$$

同様にして，$y_0 = z_0 = \dfrac{1}{4}$．よって，$G\left(\dfrac{1}{4},\dfrac{1}{4},\dfrac{1}{4}\right)$.

$$I_z = \iiint_D (x^2+y^2)\rho dxdydz = \rho\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}(x^2+y^2)dz$$

$$= \rho\int_0^1 dx\int_0^{1-x}(x^2-x^3-x^2y+(1-x)y^2-y^3)dy$$

$$= \rho\int_0^1 \left[(x^2-x^3)y-\frac{x^2}{2}y^2+\frac{1-x}{3}y^3-\frac{y^4}{4}\right]_{y=0}^{y=1-x}$$

$$= \frac{\rho}{12}\int_0^1 (7x^4-16x^3+12x^2-4x+1)dx = \frac{\rho}{12}\left[\frac{7}{5}x^5-4x^4+4x^3-2x^2+x\right]_0^1 = \frac{\rho}{30}.$$

(3) $M = \iiint_D \rho dxdydz = \rho\int_1^2 dx\int_0^{2-x} dy\int_1^{3-x-y} dz = \rho\int_1^2 dx\int_0^{2-x}(2-x-y)dy$

$$= \rho\int_1^2 \left[(2-x)y-\frac{y^2}{2}\right]_{y=0}^{y=2-x} dx = \frac{\rho}{2}\int_1^2 (2-x)^2dx = \frac{\rho}{2}\left[-\frac{(2-x)^3}{3}\right]_1^2 = \frac{\rho}{6}.$$

$$x_0 = \frac{1}{M}\iiint_D x\rho dxdydz = 6\int_1^2 dx\int_0^{2-x} dy\int_1^{3-x-y} xdz$$

$$= 6\int_1^2 dx\int_0^{2-x}(2x-x^2-yx)dy = 6\int_1^2 \left[(2x-x^2)y-\frac{x}{2}y^2\right]_{y=0}^{y=2-x} dx$$

$$= 3\int_1^2 (4x-4x^2+x^3)dx = 3\left[2x^2-\frac{4}{3}x^3+\frac{x^4}{4}\right]_1^2 = \frac{5}{4}.$$

同様にして，$z_0 = \dfrac{5}{4}$.

$$y_0 = \frac{1}{M}\iiint_D y\rho dxdydz = 6\int_1^2 dx\int_0^{2-x} dy\int_1^{3-x-y} ydz$$

$$= 6\int_1^2 dx\int_1^{2-x}\left\{(2-x)y-y^2\right\}dy = 6\int_1^2 \left[\frac{2-x}{2}y^2-\frac{y^3}{3}\right]_{y=0}^{y=2-x} dx$$

$$= \int_1^2 (2-x)^3 dx = \left[-\frac{(2-x)^4}{4} \right]_1^2 = \frac{1}{4}.$$

よって，$G\left(\frac{5}{4}, \frac{1}{4}, \frac{5}{4}\right)$.

$$\begin{aligned}
I_z &= \iiint_D (x^2+y^2)\rho dxdydz = \rho \int_1^2 dx \int_0^{2-x} dy \int_1^{3-x-y} (x^2+y^2)dz \\
&= \rho \int_1^2 dx \int_0^{2-x} \left\{ 2x^2 - x^3 - x^2 y + (2-x)y^2 - y^3 \right\} dy \\
&= \rho \int_1^2 \left[(2x^2 - x^3)y - \frac{x^2}{2}y^2 + \frac{2-x}{3}y^3 - \frac{y^4}{4} \right]_{y=0}^{y=2-x} dx \\
&= \frac{\rho}{12} \int_1^2 (7x^4 - 32x^3 + 48x^2 - 32x + 16)dx \\
&= \frac{\rho}{12} \left[\frac{7}{5}x^5 - 8x^4 + 16x^3 - 16x^2 + 16x \right]_1^2 = \frac{17}{60}\rho.
\end{aligned}$$

(4)　$E\colon x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0$ とする．

$$M = \iiint_D \rho dxdydz = \rho \int_0^1 dz \iint_E dxdy = \frac{\pi}{4}\rho.$$

$$\begin{aligned}
x_0 &= \frac{1}{M} \iiint_D x\rho dxdydz = \frac{4}{\pi} \int_0^1 dz \iint_E xdxdy = \frac{4}{\pi} \int_0^1 dy \int_0^{\sqrt{1-y^2}} xdx \\
&= \frac{4}{\pi} \int_0^1 \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \frac{2}{\pi} \int_0^1 (1-y^2)dy = \frac{2}{\pi} \left[y - \frac{y^3}{3} \right]_0^1 = \frac{4}{3\pi}.
\end{aligned}$$

同様にして，$y_0 = \frac{4}{3\pi}$．$z_0 = \frac{1}{M} \iiint_D z\rho dxdydz = \frac{4}{\pi} \int_0^1 dz \iint_E zdxdy = \int_0^1 zdz = \frac{1}{2}$．よって，$G\left(\frac{4}{3\pi}, \frac{4}{3\pi}, \frac{1}{2}\right)$．極座標変換より，

$$I_z = \iiint_D (x^2+y^2)\rho dxdydz = \rho \int_0^1 dz \iint_E (x^2+y^2)dxdy = \rho \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 dr = \frac{\pi}{8}\rho.$$

(5)　$x = u-1$，$v = v+1$，$z = w$ により D は $E\colon u^2+v^2 \leqq 1,\ 0 \leqq w \leqq 1$ に写る．$F\colon u^2+v^2 \leqq 1$ とおく．

$$M = \iiint_D \rho dxdydz = \rho \iiint_E dudvdw = \rho \int_0^1 dw \iint_F dudv = \pi\rho.$$

$x_0 = \frac{1}{M} \iiint_E (u-1)\rho dudvdw = -1$．同様にして，$y_0 = 1$．

$$z_0 = \frac{1}{M} \iiint_D z\rho dzdydz = \frac{1}{\pi} \int_0^1 dw \iint_F wdudv = \frac{1}{2}.$$

よって，$G\left(-1, 1, \frac{1}{2}\right)$．極座標変換より，

$$\begin{aligned}
I_z &= \iiint_D (x^2+y^2)\rho dxdydz = \rho \int_0^1 dw \iint_F \left\{ (u-1)^2 + (v+1)^2 \right\} dudv \\
&= \rho \int_0^{2\pi} d\theta \int_0^1 \left\{ r^3 - 2r^2(\cos\theta + \sin\theta) + 2r \right\} dr = \rho \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^3}{3}(\cos\theta + \sin\theta) + r^2 \right]_{r=0}^{r=1} d\theta
\end{aligned}$$

$$= \rho \int_0^{2\pi} \left\{ \frac{5}{4} - \frac{1}{3}(\cos\theta + \sin\theta) \right\} d\theta = \frac{5}{2}\pi\rho.$$

(6)　極座標変換より，

$$M = \iiint_D \rho dxdydz = \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2 \sin\theta dr$$
$$= \frac{\pi}{2}\rho \left(\int_0^{\frac{\pi}{2}} \sin\theta d\theta \right) \left(\int_0^1 r^2 dr \right) = \frac{\pi}{2}\rho \left[-\cos\theta \right]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^1 = \frac{\pi}{6}\rho.$$
$$x_0 = \frac{1}{M} \iiint_D x\rho dxdydz = \frac{6}{\pi} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 \sin^2\theta \cos\varphi dr$$
$$= \frac{6}{\pi} \left(\int_0^{\frac{\pi}{2}} \cos\varphi d\varphi \right) \left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta \right) \left(\int_0^1 r^3 dr \right)$$
$$= \frac{3}{\pi} \left[\sin\varphi \right]_0^{\frac{\pi}{2}} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^1 = \frac{3}{8}.$$

同様にして，$y_0 = z_0 = \dfrac{3}{8}$．よって，$G\left(\dfrac{3}{8}, \dfrac{3}{8}, \dfrac{3}{8} \right)$.

$$I_z = \iiint_D (x^2+y^2)\rho dxdydz = \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^4 \sin^3\theta dr$$
$$= \frac{\pi}{2}\rho \left(\int_0^{\frac{\pi}{2}} \frac{3\sin\theta - \sin 3\theta}{4} d\theta \right) \left(\int_0^1 r^4 dr \right) = \frac{\pi}{8}\rho \left[-3\cos\theta + \frac{\cos 3\theta}{3} \right]_0^{\frac{\pi}{2}} \left[\frac{r^5}{5} \right]_0^1 = \frac{\pi}{15}\rho.$$

(7)　D は $(-1,-1,1)$ を中心とする半径 1 の球なので，$G(-1,-1,1)$．$x = u-1$，$y = v-1$，$z = w+1$ により，D は E: $u^2+v^2+z^2 \leqq 1$ に写るので，極座標変換より，

$$I_z = \iiint_D (x^2+y^2)\rho dxdydz = \rho \iiint_E \left\{ (u-1)^2 + (v-1)^2 \right\} dudvdw$$
$$= \rho \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \left\{ r^4 \sin^3\theta - 2r^3 \sin^2\theta (\cos\varphi + \sin\varphi) + 2r^2 \sin\theta \right\} dr$$
$$= \rho \int_0^1 dr \int_0^{\pi} \left[(r^4 \sin^3\theta + 2r^2 \sin\theta)\varphi - 2r^3 \sin^2\theta (\sin\varphi - \cos\varphi) \right]_{\varphi=0}^{\varphi=2\pi} d\theta$$
$$= 2\pi\rho \int_0^1 dr \int_0^{\pi} \left(r^4 \frac{3\sin\theta - \sin 3\theta}{4} + 2r^2 \sin\theta \right) d\theta$$
$$= 2\pi\rho \int_0^1 \left[r^4 \left(-3\frac{\cos\theta}{4} + \frac{\cos 3\theta}{12} \right) - 2r^2 \cos\theta \right]_{\theta=0}^{\theta=\pi} dr$$
$$= 2\pi\rho \int_0^1 \left(\frac{4}{3} r^4 + 4r^2 \right) dr = \frac{16}{5}\pi\rho.$$

(8)　$x = r\cos\theta$，$y = r\sin\theta$ とすると，

$$M = \iiint_D \rho dxdydz = \rho \int_0^1 dz \int_0^{2\pi} d\theta \int_0^{1-z} r dr = \rho \int_0^1 dz \int_0^{2\pi} \frac{(1-z)^2}{2} d\theta$$
$$= \pi\rho \int_0^1 (1-z)^2 dz = \pi\rho \left[-\frac{(1-z)^3}{3} \right]_0^1 = \frac{\pi}{3}\rho.$$

対称性より，$x_0 = y_0 = 0$.

$$z_0 = \frac{1}{M} \iiint_D z\rho dxdydz = \frac{3}{\pi} \int_0^1 dz \int_0^{2\pi} d\theta \int_0^{1-z} rz dr = \frac{3}{\pi} \int_0^1 dz \int_0^{2\pi} z \frac{(1-z)^2}{2} d\theta$$

$$= 3\int_0^1 (z - 2z^2 + z^3)dz = 3\left[\frac{z^2}{2} - \frac{2}{3}z^3 + \frac{z^4}{4}\right]_0^1 = \frac{1}{4}.$$

よって，$G\left(0, 0, \frac{1}{4}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2)\rho dzdydz = \rho\int_0^1 dz\int_0^{2\pi} d\theta\int_0^{1-z} r^3 dr \\ &= 2\pi\rho\int_0^1 \frac{(1-z)^4}{4}dz = \frac{\rho}{2}\left[-\frac{(1-z)^5}{5}\right]_0^1 = \frac{\pi}{10}\rho. \end{aligned}$$

(9)　$x = 2r\sin\theta\cos\varphi$，$y = \frac{1}{2}r\sin\theta\sin\varphi$，$z = r\cos\theta$ と変換して，

$$\begin{aligned} M &= \iiint_D \rho dxdydz = \rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\int_0^1 r^2\sin\theta dr \\ &= \pi\rho\left(\int_0^{\frac{\pi}{2}}\sin\theta d\theta\right)\left(\int_0^1 r^2 dr\right) = \pi\rho\left[-\cos\theta\right]_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^1 = \frac{\pi}{3}\rho. \\ x_0 &= \frac{1}{M}\iiint_D x\rho dxdydz = \frac{3}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\int_0^1 2r^3\sin^2\theta\cos\varphi dr \\ &= \frac{6}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi d\varphi\right)\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}d\theta\right)\left(\int_0^1 r^3 dr\right) \\ &= \frac{3}{\pi}\left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^1 = \frac{3}{4}. \end{aligned}$$

対称性より，$y_0 = 0$.

$$\begin{aligned} z_0 &= \frac{1}{M}\iiint_D z\rho dxdydz = \frac{3}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\int_0^1 r^3\sin\theta\cos\theta dr \\ &= 3\left(\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}d\theta\right)\left(\int_0^1 r^3 dr\right) = \frac{3}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^1 = \frac{3}{8}. \end{aligned}$$

よって，$G\left(\frac{3}{4}, 0, \frac{3}{8}\right)$.

$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2)\rho dxdydz = \rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\int_0^1\left(4\cos^2\varphi + \frac{1}{4}\sin^2\varphi\right)r^4\sin^3\theta dr \\ &= \rho\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{2(\cos 2\varphi + 1) + \frac{1-\cos 2\varphi}{8}\right\}d\varphi\right)\left(\int_0^{\frac{\pi}{2}}\sin\theta d\theta\right)\left(\int_0^1 r^4 dr\right) \\ &= \rho\left[\sin 2\varphi + \frac{17}{8}\varphi - \frac{\sin 2\varphi}{16}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-\cos\theta\right]_0^{\frac{\pi}{2}}\left[\frac{r^5}{5}\right]_0^1 = \frac{17}{80}\pi\rho. \end{aligned}$$