5　重積分

5.1　2 重積分

問 1　(1)　$\displaystyle \iint_D (\cos^2 x)y^3 dxdy = \left(\int_{-\pi}^{\pi} \cos^2 x dx\right)\left(\int_{-3}^{1} y^3 dy\right) = \int_0^{\pi} (\cos 2x + 1)dx \left[\frac{x^4}{4}\right]_{-3}^{1}$

$\displaystyle = -20\left[\frac{\sin 2x}{2} + x\right]_0^{\pi} = -20\pi$

(2)　$\displaystyle \iint_D (x^4 - y^3)dxdy = 12\int_{-4}^{3} x^4 dx - 7\int_{-5}^{7} y^3 dy = 12\left[\frac{x^5}{5}\right]_{-4}^{3} - 7\left[\frac{y^4}{4}\right]_{-5}^{7}$

$\displaystyle = 12\cdot\frac{1267}{5} - 7\cdot 444 = -\frac{336}{5}$

(3)　$\displaystyle \iint_D (x^2 - y^2)^3 dxdy = \iint_D (x^6 - 3x4y^2 + 3x^2y^4 - y^6)dxdy$

$\displaystyle = 4\int_{-2}^{2} x^6 dx - 3\int_{-2}^{2} x^4 dx\int_{-2}^{2} y^2 dy + 3\int_{-2}^{2} x^2 dx\int_{-2}^{2} y^4 dy - 4\int_{-2}^{2} y^6 dy = 0$

(4)　$\displaystyle \iint_D \sin^2(x+y)dxdy = \iint_D \frac{1-\cos 2(x+y)}{2}dxdy$

$\displaystyle = \frac{1}{2}\left(\frac{3}{4}\pi^2 - \int_0^{\frac{3\pi}{2}}\left[\frac{\sin 2(x+y)}{2}\right]_{y=0}^{y=\frac{\pi}{2}} dx\right) = \frac{1}{2}\left(\frac{3}{4}\pi^2 + \int_0^{\frac{3\pi}{2}} \sin 2x dx\right)$

$\displaystyle = \frac{1}{2}\left(\frac{3}{4}\pi^2 + \left[-\frac{\cos 2x}{2}\right]_0^{\frac{3\pi}{2}}\right) = \frac{3\pi^2 + 4}{8}$

問 2

(1)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq x(1-x^2)^{\frac{1}{4}}$ とおくと，対称性より，求める面積は $\displaystyle 4|D| = 4\iint_D dxdy = 4\int_0^1 x(1-x^2)^{\frac{1}{4}}dx$．ここで，$z = 1 - x^2$ とおくと $\dfrac{dz}{dx} = -2x$ なので，

$$4|D| = 2\int_0^1 z^{\frac{1}{4}}dz = 2\left[\frac{4}{5}z^{\frac{5}{4}}\right]_0^1 = \frac{8}{5}.$$

(2)　交点は $(x,y) = (0,0),\ \left(\dfrac{1}{\sqrt[3]{18}}, \dfrac{1}{\sqrt[3]{12}}\right)$ なので，$D\colon 0 \leqq x \leqq \dfrac{1}{\sqrt[3]{18}},\ 3x^2 \leqq y \leqq \sqrt{\dfrac{x}{2}}$ とおくと，求める面積は

$$\iint_D dxdy = \int_0^{\frac{1}{\sqrt[3]{18}}}\left(\sqrt{\frac{x}{2}} - 3x^2\right)dx = \left[\frac{\sqrt{2}}{3}x^{\frac{3}{2}} - x^3\right]_0^{\frac{1}{\sqrt[3]{18}}} = \frac{1}{18}.$$

(3)　$D\colon 0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \sin^4 x$ とおく．$\sin^4 x = \dfrac{3 - \cos 2x + \cos 4x}{8}$ なので，対称性より，求める面積は

$$2\iint_D dxdy = \frac{1}{4}\int_0^{\pi}(3 - 4\cos 2x + \cos 4x)dx = \frac{1}{4}\left[3\theta - \sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi} = \frac{3}{4}\pi.$$

(4)　$D\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq (1-\sqrt{x})^2$ とおくと，対称性より，求める面積は

$$4\iint_D dxdy = 4\int_0^1 \left(1-\sqrt{x}\right)^2 dx = 4\int_0^1 \left(1 - 2\sqrt{x} + x\right)dx = 4\left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3}.$$

(5)　y について解くと $y=\dfrac{-x\pm\sqrt{-3x^2+4}}{2}$ なので，D: $-\dfrac{2}{\sqrt{3}}\leqq x\leqq\dfrac{2}{\sqrt{3}}$, $\dfrac{-x-\sqrt{-3x^2+4}}{2}\leqq y\leqq\dfrac{-x+\sqrt{-3x^2+4}}{2}$ とおくと，求める面積は，

$$\iint_D dxdy=\int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}\sqrt{4-3x^2}dx=2\int_0^{\frac{2}{\sqrt{3}}}\sqrt{4-3x^2}dx$$
$$=2\left[\frac{1}{2}x\sqrt{4-3x^2}+\frac{2}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}}{2}x\right]_0^{\frac{2}{\sqrt{3}}}=\frac{2}{\sqrt{3}}\pi.$$

(6)　D: $0\leqq x\leqq 1$, $0\leqq y\leqq\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ とおくと，対称性より，求める面積は $4|D|=4\iint_D dxdy=4\int_0^1\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}dx$．ここで，$x=\sin^3\theta$ とおくと $\dfrac{dx}{d\theta}=3\sin^2\theta\cos\theta$ であり，$\cos^4\theta=\dfrac{1}{8}(3+4\cos 2x+\cos 4\theta)$，$\cos^6\theta=\dfrac{1}{32}(10+15\cos 2\theta+6\cos 4\theta+\cos 6\theta)$ より，

$$4|D|=12\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta d\theta=12\int_0^{\frac{\pi}{2}}\left(\cos^4\theta-\cos^6\theta\right)d\theta$$
$$=\frac{3}{8}\int_0^{\frac{\pi}{2}}(2+16\cos 2\theta-2\cos 4\theta-\cos 6\theta)d\theta$$
$$=\frac{3}{8}\left[2\theta+8\sin 2\theta-\frac{1}{2}\sin 4\theta-\frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{2}}=\frac{3}{8}\pi.$$

問 3　(1)　$\displaystyle\iint_D\sin(x+y)dxdy=\int_0^\pi dx\int_0^{-x+\pi}\sin(x+y)dy$

$$=\int_0^\pi\left[-\cos(x+y)\right]_{y=0}^{y=-x+\pi}dx=\int_0^\pi(1+\cos x)dx=[x+\sin x]_0^\pi=\pi$$

(2)　$\displaystyle\iint_D|\cos x\sin y|dxdy=-4\int_{\frac{\pi}{2}}^\pi dx\int_0^{-x+\pi}\cos x\sin ydy$

$$=-4\int_{\frac{\pi}{2}}^\pi\cos x\left[-\cos y\right]_0^{-x+\pi}dx=-4\int_{\frac{\pi}{2}}^\pi\cos(\cos^2x+\cos x)dx$$
$$=-2\int_{\frac{\pi}{2}}^\pi(\cos 2x+1+2\cos x)dx=-2\left[\frac{\sin 2x}{2}+x+2\sin x\right]_{\frac{\pi}{2}}^\pi=4-\pi$$

(3)　$\displaystyle\iint_D|xy|^2dxdy=4\int_0^{\sqrt[3]{2}}dx\int_0^{\sqrt[3]{2-x^3}}x^2y^2dy=4\int_0^{\sqrt[3]{2}}x^2\left[\frac{y^3}{3}\right]_0^{\sqrt[3]{2-x^3}}dx$

$$=\frac{4}{3}\int_0^{\sqrt[3]{2}}x^2(2-x^3)dx=\frac{4}{3}\left[\frac{2}{3}x^3-\frac{x^6}{6}\right]_0^{\sqrt[3]{2}}=\frac{8}{9}$$

(4)　$\displaystyle\iint_D e^{\frac{x}{y}}dxdy=\int_0^1dy\int_0^{y^2}e^{\frac{x}{y}}dx=\int_0^1\left[ye^{\frac{x}{y}}\right]_{x=0}^{x=y^2}dy=\int_0^1y\left(e^y-1\right)dy$

$$=[ye^y]_0^1-\int_0^1e^ydy-\left[\frac{y^2}{2}\right]_0^1=\frac{1}{2}$$

(5)　$\displaystyle\iint_D\log\frac{x}{y^2}dxdy=\int_2^5dx\int_1^x(\log x-2\log y)dy$

$$=\int_2^5(x-1)\log xdx-2\int_2^5[y\log y-y]_1^xdx=-\int_2^5x\log xdx+\int_2^5(-\log x+2x-2)dx$$
$$=-\left[\frac{x^2}{2}\log x\right]_2^5+\frac{1}{2}\int_2^5xdx+[-x\log x+x+x^2-2x]_2^5$$

$$= -\frac{35}{2}\log 5 + 4\log 2 + 18 + \frac{1}{2}\left[\frac{x^2}{2}\right]_2^5 = -\frac{35}{2}\log 5 + 4\log 2 + \frac{93}{4}$$

(6) $\displaystyle \iint_D xy^2 dxdy = 2\int_3^6 dx \int_{\sqrt{x}}^{\sqrt{6}} xy^2 dy = 2\int_3^6 x\left[\frac{y^3}{3}\right]_{\sqrt{x}}^{\sqrt{6}} dx$

$$= \frac{2}{3}\int_3^6 \left(6^{\frac{3}{2}}x - x^{\frac{3}{2}}\right)dx = \frac{2}{3}\left[\frac{6^{\frac{3}{2}}}{2}x^2 - \frac{2}{7}x^{\frac{7}{2}}\right]_3^6 = \frac{18}{7}\left(2\sqrt{3} + 5\sqrt{6}\right)$$

(7) $\displaystyle \iint_D xydxdy = \int_0^1 dx \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} xydy = 0$

(8) $\displaystyle \iint_D \sqrt{y-x^2}dxdy = \int_{-2}^1 dx \int_{x^2}^{-x+2} \sqrt{y-x^2}dy$

$$= \int_{-2}^1 \left[\frac{2}{3}\left(y-x^2\right)^{\frac{3}{2}}\right]_{y=x^2}^{y=-x+2} dx = \frac{2}{3}\int_{-2}^1 \left(-x^2-x+2\right)^{\frac{3}{2}}dx$$

$$= \frac{2}{3}\int_{-2}^1 \left\{-\left(x+\frac{1}{2}\right)^2 + \frac{9}{4}\right\}^{\frac{3}{2}} dx = \frac{2}{3}\int_{-\frac{3}{2}}^{\frac{3}{2}} \left(-x^2+\frac{9}{4}\right)^{\frac{3}{2}} dx$$

$$= \frac{4}{3}\int_0^{\frac{3}{2}} \left(-x^2+\frac{9}{4}\right)^{\frac{3}{2}} dx = \frac{27}{4}\int_0^1 \left(-x^2+1\right)^{\frac{3}{2}} dx.$$

ここで，

$$I = \int_0^1 \left(-x^2+1\right)^{\frac{3}{2}} dx = \left[x\left(-x^2+1\right)^{\frac{3}{2}}\right]_0^1 + 3\int_0^1 x^2\left(-x^2+1\right)^{\frac{1}{2}} dx$$

$$= -3I + 3\int_0^1 \sqrt{1-x^2}dx = -3I + 3\left[\frac{x\sqrt{1-x^2}+\sin^{-1}x}{2}\right]_0^1 = -3I + \frac{3}{4}\pi$$

なので，$I = \dfrac{3}{16}\pi$．よって，求める値は $\dfrac{81}{64}\pi$．