3　積分

3.1　不定積分
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演習問題 3

3.1 不定積分

問 1. (1) $\displaystyle\int \frac{dx}{\cos^2(3x+1)} = \frac{1}{3}\tan(3x+1)$

(2) $\displaystyle\int \frac{dx}{\sqrt{4-(2x+1)^2}} = \frac{1}{2}\sin^{-1}\frac{2x+1}{2} = \frac{1}{2}\sin^{-1}\left(x+\frac{1}{2}\right)$

(3) $\displaystyle\int \frac{dx}{x^2+4x+6} = \int \frac{dx}{(x+2)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+2}{\sqrt{2}}$

(4) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(\cos x)'}{\cos x}\,dx = -\log|\cos x|$

(5) $\displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx = \int \frac{(x+\sin x)'}{x+\sin x}\,dx = \log|x+\sin x|$

(6) $\displaystyle\int x^2(x^3+1)^{\frac{5}{2}}dx = \frac{1}{3}\int (x^3+1)^{\frac{5}{2}}(x^3+1)'dx = \frac{1}{3}\cdot\frac{1}{\frac{5}{2}+1}(x^3+1)^{\frac{5}{2}+1} = \frac{2}{21}(x^3+1)^{\frac{7}{2}}$

(7)
$$\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int (x^2+2x+3)^{-\frac{1}{2}}(x^2+2x+3)'\,dx$$
$$= \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}(x^2+2x+3)^{-\frac{1}{2}+1} = \sqrt{x^2+2x+3}$$

(8) $\displaystyle\int \frac{(\log x)^7}{x}\,dx = \int (\log x)^7(\log x)'\,dx = \frac{1}{8}(\log x)^8$

(9)
$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}\left\{(1-x^2)\right\}'\,dx = -\frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}(1-x^2)^{-\frac{1}{2}+1}$$
$$= -\sqrt{1-x^2}$$

問 2. (1) $\displaystyle\int (x-3)(2x-1)\,dx = \int (2x^2-7x+3)\,dx = \frac{2}{3}x^3 - \frac{7}{2}x^2 + 3x.$

(2) $\displaystyle\int \left(x+\frac{1}{x}\right)^2 dx = \int \left(x^2+2+\frac{1}{x^2}\right)dx = \frac{1}{3}x^3 + 2x - \frac{1}{x}.$

(3)
$$\int \frac{(x-1)(\sqrt{x}-2)}{x^2}\,dx = \int \frac{x^{\frac{3}{2}}-2x-x^{\frac{1}{2}}+2}{x^2}\,dx = \int \left(x^{-\frac{1}{2}} - \frac{2}{x} - x^{-\frac{3}{2}} + \frac{2}{x^2}\right)dx$$
$$= 2\sqrt{x} - 2\log|x| + 2x^{-\frac{1}{2}} - \frac{2}{x} = 2\left(\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{x} - \log x\right)$$

(4) $\displaystyle\int \left(e^x+e^{-x}\right)^2 dx = \int \left(e^{2x}+2+e^{-2x}\right)dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} = \sinh 2x + 2x$

(5) $\displaystyle\int \left(2^{x+1}+3^{x+1}\right)dx = \frac{2^{x+1}}{\log 2} + \frac{3^{x+1}}{\log 3}$

(6) $\displaystyle\int \frac{dx}{\cos^2 x\sin^2 x} = 4\int \frac{dx}{\sin^2 2x} = 4\cdot\left(\frac{1}{2}\right)\cdot(-\cot 2x) = -2\cot 2x$

別解. $\displaystyle\int \frac{dx}{\cos^2 x\sin^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\cos^2 x\sin^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx = \tan x - \cot x$

(7) $\displaystyle\int \cos^2\frac{x}{2}\,dx = \int \frac{1}{2}(1+\cos x)\,dx = \frac{1}{2}(x+\sin x)$

(8) $\displaystyle\int \sin^3 x\,dx = \int \sin x\cdot\sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx = \int \sin x\,dx - \int \sin x\cos^2 x\,dx$

$$= -\cos x + \frac{1}{3}\cos^3 x$$

別解. $\sin 3x = 3\sin x - 4\sin^3 x$ より,

$$\int \sin^3 x\,dx = \frac{1}{4}\int (3\sin x - \sin 3x)\,dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x.$$

(9) $\displaystyle\int \sin x\cos 2x\,dx = \int \sin x(2\cos^2 x - 1)\,dx = 2\int \sin x\cos^2 x\,dx - \int \sin x\,dx$

$$= -\frac{2}{3}\cos^3 x + \cos x$$

別解. $2\sin x\cos 2x = \sin(x+2x) + \sin(x-2x)$ より,

$$\int \sin x\cos 2x\,dx = \frac{1}{2}\int (\sin 3x - \sin x)\,dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x.$$

問 3. (1) $x = \sin t$ $\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$ とおくと,$dx = \cos t dt$,$\cos t > 0$.よって,$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t$ より,

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos t}{\sin^2 t\cos t}\,dt = \int \frac{dt}{\sin^2 t} = -\cot t = -\frac{\cos t}{\sin t} = -\frac{\sqrt{1-x^2}}{x}.$$

(2) $x = \tan t$ $\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$ とおくと,$dx = \dfrac{dt}{\cos^2 t}$,$\cos t > 0$.よって

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{1}{(1+\tan^2 t)^{\frac{3}{2}}}\cdot\frac{dt}{\cos^2 t} = \int \cos^3 t\cdot\frac{dt}{\cos^2 t} = \int \cos t\,dt$$

$$= \sin t = \tan t\cos t = \frac{\tan t}{\sqrt{1+\tan^2 t}} = \frac{x}{\sqrt{1+x^2}}.$$

(3) $x^2 = t$ とおくと,$x\,dx = \frac{1}{2}\,dt$.よって

$$\int \frac{x}{\sqrt{x^4-2}}\,dx = \int \frac{1}{\sqrt{t^2-2}}\cdot\frac{1}{2}\,dt = \frac{1}{2}\log\left|t+\sqrt{t^2-2}\right| = \frac{1}{2}\log\left(x^2+\sqrt{x^4-2}\right).$$

(4) $\sqrt{1+x^2} = t$ とおくと,$x^2 = t^2 - 1$,$x\,dx = t\,dt$.よって

$$\int x^3\sqrt{1+x^2}\,dx = \int (t^2-1)t\cdot t\,dt = \int (t^4-t^2)\,dt = \frac{1}{5}t^5 - \frac{1}{3}t^3$$

$$= \frac{1}{15}(3t^2-5)t^3 = \frac{1}{15}(3x^2-2)(x^2+1)\sqrt{x^2+1}.$$

参考. $\displaystyle\int x^3\sqrt{1+x^2}\,dx = \frac{1}{2}\int (1+x^2)'\cdot x^2\sqrt{1+x^2}\,dx$

$$= \frac{1}{2}\int (1+x^2)'\left\{(1+x^2)-1\right\}\sqrt{1+x^2}\,dx$$

$$= \frac{1}{2}\int (1+x^2)'\left\{(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}\right\}dx$$

$$= \frac{1}{2}\left\{\frac{2}{5}(1+x^2)^{\frac{5}{2}} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}\right\}$$
$$= \frac{1}{15}(3x^2-2)(x^2+1)\sqrt{x^2+1}.$$

(5)　$t = \tan x \ \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$ とおくと，$dt = \frac{dx}{\cos^2 x}$，$\cos x > 0$．よって

$$\int \frac{dx}{\cos^2 x + 3\sin^2 x} = \int \frac{\cos^2 x}{\cos^2 x\,(1+3\tan^2 x)}\,dt = \int \frac{dt}{1+3t^2} = \frac{1}{3}\int \frac{dt}{t^2+\frac{1}{3}}$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}t = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$$

(6)　$\sqrt{x^2-1} = t - x$ とおくと，

$$x = \frac{t^2+1}{2t}, \quad dx = \frac{t^2-1}{2t^2}\,dt, \quad \sqrt{x^2-1} = \frac{t^2-1}{2t}.$$

$$\therefore \quad \int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{4t^2}{(t^2+1)^2}\cdot\frac{2t}{t^2-1}\cdot\frac{t^2-1}{2t^2}\,dt = 4\int \frac{t}{(t^2+1)^2}\,dt$$
$$= -\frac{2}{t^2+1} = -\frac{2}{(\sqrt{x^2-1}+x)^2+1}$$
$$= -\frac{1}{x(\sqrt{x^2-1}+x)} = \frac{\sqrt{x^2-1}-x}{x} = \frac{\sqrt{x^2-1}}{x} - 1.$$

参考.　$\sqrt{x^2-1} = t + x$ とおくと，

$$x = -\frac{t^2+1}{2t}, \quad dx = -\frac{t^2-1}{2t^2}\,dt, \quad \sqrt{x^2-1} = \frac{t^2-1}{2t}.$$

$$\therefore \quad \int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{4t^2}{(t^2+1)^2}\cdot\frac{2t}{t^2-1}\cdot\left(-\frac{t^2-1}{2t^2}\right)dt = -4\int \frac{t}{(t^2+1)^2}\,dt$$
$$= \frac{2}{t^2+1} = \frac{2}{(\sqrt{x^2-1}-x)^2+1}$$
$$= \frac{1}{x(x-\sqrt{x^2-1})} = \frac{x+\sqrt{x^2-1}}{x} = \frac{\sqrt{x^2-1}}{x} + 1.$$

参考.　$\frac{1}{x^2} = t$ とおくと，$x = \pm\frac{1}{\sqrt{t}}$．$x > 1$ のときは，$x = \frac{1}{\sqrt{t}}$ なので，$dx = -\frac{1}{2}t^{-\frac{3}{2}}dt$．よって

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{t}{\sqrt{\frac{1}{t}-1}}\left(-\frac{1}{2t\sqrt{t}}\right)dt = -\frac{1}{2}\int \frac{dt}{\sqrt{1-t}} = \sqrt{1-t}$$
$$= \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{\sqrt{x^2}} = \frac{\sqrt{x^2-1}}{x}.$$

$x < -1$ のときは，$x = -\frac{1}{\sqrt{t}}$ なので，$dx = \frac{1}{2}t^{-\frac{3}{2}}dt$．よって

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{t}{\sqrt{\frac{1}{t}-1}}\cdot\frac{dt}{2t\sqrt{t}} = \frac{1}{2}\int \frac{dt}{\sqrt{1-t}} = -\sqrt{1-t}$$

$$= -\sqrt{1-\frac{1}{x^2}} = -\frac{\sqrt{x^2-1}}{\sqrt{x^2}} = -\frac{\sqrt{x^2-1}}{-x} = \sqrt{1-\frac{1}{x^2}}.$$

以上より

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \sqrt{1-\frac{1}{x^2}}.$$

問 4. $I = \displaystyle\int \sqrt{a^2-x^2}\,dx$ とおくと，部分積分法より

$$\begin{aligned}
I &= x\sqrt{a^2-x^2} - \int x\cdot\frac{1}{2}\left(a^2-x^2\right)^{-\frac{1}{2}}(-2x)\,dx \\
&= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx \\
&= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx \\
&= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}} \\
&= x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\frac{x}{a}
\end{aligned}$$

となる．よって，上式を I について解けばよい．

問 5. (1) $\displaystyle\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx = x^2e^x - 2xe^x + 2\int e^x\,dx = (x^2-2x+2)e^x$

(2) $\displaystyle\int (x-2)\cos 3x\,dx = \frac{1}{3}(x-2)\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{1}{3}(x-2)\sin 3x + \frac{1}{9}\cos 3x$

(3) $I = \displaystyle\int e^x\cos x\,dx$ とおくと，

$$I = e^x\cos x + \int e^x\sin x\,dx = e^x(\cos x+\sin x) - \int e^x\cos x\,dx = e^x(\cos x + \sin x) - I$$

$$\therefore\quad \int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x+\cos x).$$

(4) $\displaystyle\int \frac{\log x}{x^2}\,dx = -\frac{\log x}{x} + \int\frac{dx}{x^2} = -\frac{\log x}{x} - \frac{1}{x} = -\frac{\log x + 1}{x}$

(5)
$$\begin{aligned}
\int \log(x^2+1)\,dx &= x\log(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx \\
&= x\log(x^2+1) - 2\int\frac{(x^2+1)-1}{x^2+1}\,dx \\
&= x\log(x^2+1) - 2\int\left(1-\frac{1}{x^2+1}\right)dx \\
&= x\log(x^2+1) + 2\tan^{-1}x - 2x
\end{aligned}$$

(6)
$$\begin{aligned}
\int (\log x)^2\,dx &= x(\log x)^2 - \int x\cdot\frac{2}{x}\log x\,dx = x(\log x)^2 - 2\int\log x\,dx \\
&= x(\log x)^2 - 2x\log x + 2x = x\Big\{(\log x)^2 - 2\log x + 2\Big\}
\end{aligned}$$

(7) $\displaystyle\int (x+1)e^x\log x\,dx = \int e^x x\log x\,dx + \int e^x\log x\,dx$

$$
\begin{aligned}
&= e^x x\log x - \int e^x\left(\log x + \frac{x}{x}\right)dx + \int e^x \log x\,dx \\
&= xe^x \log x - \int e^x \log x\,dx - \int e^x dx + \int e^x \log x\,dx \\
&= xe^x \log x - e^x = e^x\,(x\log x - 1)
\end{aligned}
$$

(8) $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2}$

(9) $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{x^2+1}\,dx = x\tan^{-1} x - \frac{1}{2}\log(x^2+1)$

問 6. I と J をそれぞれ部分積分すると，

$$
I = \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}J
$$

$$
\therefore\quad bI - aJ = -e^{ax}\cos bx \tag{1}
$$

$$
J = \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I
$$

$$
\therefore\quad aI + bJ = e^{ax}\sin bx \tag{2}
$$

$(2)\times a + (1)\times b$ より，

$$
(a^2+b^2)I = e^{ax}(a\sin bx - b\cos bx).
$$

また，$(2)\times b - (1)\times a$ より，

$$
(a^2+b^2)J = e^{ax}(b\sin bx + a\cos bx).
$$

ゆえに

$$
I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx),\quad J = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).
$$

問 7. 部分積分法を用いて計算すると，

$$
\begin{aligned}
A\,I_n &= \int \frac{A}{(x^2+A)^n}\,dx = \int \frac{(x^2+A)-x^2}{(x^2+A)^n}\,dx \\
&= \int \frac{dx}{(x^2+A)^{n-1}} - \int \frac{x}{(x^2+A)^n}\cdot x\,dx \\
&= I_{n-1} + \frac{1}{2(n-1)}\left\{\frac{1}{(x^2+A)^{n-1}}\cdot x - \int \frac{dx}{(x^2+A)^{n-1}}\right\} \\
&= \frac{1}{2(n-1)}\left\{\frac{x}{(x^2+A)^{n-1}} + (2n-3)I_{n-1}\right\}.
\end{aligned}
$$

よって

$$
I_n = \frac{1}{2(n-1)A}\left\{\frac{x}{(x^2+A)^{n-1}} + (2n-3)I_{n-1}\right\}\quad (n \geqq 2)
$$

次に，上で示した漸化式より，

$$
\begin{aligned}
\int \frac{dx}{(x^2+1)^2} = I_2 &= \frac{1}{2}\left(\frac{x}{x^2+1} + I_1\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right) \\
&= \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1} x\right).
\end{aligned}
$$

参考．部分積分法を用いて計算すると，

$$\begin{aligned}\int \frac{dx}{x^2+1} &= \int x' \frac{dx}{x^2+1} = \frac{x}{x^2+1} - \int x \left(\frac{1}{x^2+1}\right)' dx \\ &= \frac{x}{x^2+1} + \int \frac{2x^2}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + \int \frac{2(x^2+1)-2}{(x^2+1)^2}\,dx \\ &= \frac{x}{x^2+1} + 2\int \frac{dx}{x^2+1} - 2\int \frac{dx}{(x^2+1)^2}.\end{aligned}$$

$$\therefore \quad \int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1} x\right).$$

参考．$n=1$ のときを考える．$A>0$ のときは,

$$I_1 = \int \frac{dx}{x^2+(\sqrt{A})^2} = \frac{1}{\sqrt{A}} \tan^{-1} \frac{x}{\sqrt{A}}.$$

$A<0$ のときは,

$$\begin{aligned}I_1 &= \int \frac{dx}{x^2-(\sqrt{-A})^2} = \frac{1}{2\sqrt{-A}} \int \left(\frac{1}{x-\sqrt{-A}} - \frac{1}{x+\sqrt{-A}}\right) dx \\ &= \frac{1}{2\sqrt{-A}} \log\left|\frac{x-\sqrt{-A}}{x+\sqrt{-A}}\right|.\end{aligned}$$

$$\therefore \quad I_1 = \begin{cases} \dfrac{1}{\sqrt{A}} \tan^{-1} \dfrac{x}{\sqrt{A}} & (A>0) \\ \dfrac{1}{2\sqrt{-A}} \log\left|\dfrac{x-\sqrt{-A}}{x+\sqrt{-A}}\right| & (A<0). \end{cases}$$

問 8. (1)

$$\begin{aligned}\int \frac{dx}{(x+1)(x+3)} &= \frac{1}{2}\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx = \frac{1}{2}\left(\log|x+1| - \log|x+3|\right) \\ &= \frac{1}{2}\log\left|\frac{x+1}{x+3}\right|\end{aligned}$$

(2) $\dfrac{3x^2-x+1}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{(x-1)^2}$ とおくと，

$$3x^2-x+1 = (A+B)x^2 + (-2A+C)x + A.$$

上式の両辺の係数を比較すると， $A+B=3$，$-2A+C=-1$，$A=1$． よって，$A=C=1$，$B=2$．ゆえに

$$\frac{3x^2-x+1}{x(x-1)^2} = \frac{1}{x} + \frac{2x+1}{(x-1)^2} = \frac{1}{x} + \frac{2(x-1)+3}{(x-1)^2} = \frac{1}{x} + \frac{2}{x-1} + \frac{3}{(x-1)^2}.$$

よって

$$\begin{aligned}\int \frac{3x^2-x+1}{x(x-1)^2}\,dx &= \int \left\{\frac{1}{x} + \frac{2}{x-1} + \frac{3}{(x-1)^2}\right\} dx \\ &= \log|x| + 2\log|x-1| - \frac{3}{x-1} \\ &= \log\left|x(x-1)^2\right| - \frac{3}{x-1}.\end{aligned}$$

(3) $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$ とおくと，

$$1 = (A+B)x^2 + (-A+B+C)x + A + C.$$

上式の両辺の係数を比較すると，$A+B=0$，$-A+B+C=0$，$A+C=1$．よって，$A=-B=\dfrac{1}{3}$，$C=\dfrac{2}{3}$．ゆえに

$$\frac{1}{x^3+1} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right).$$

よって

$$\begin{aligned}
\int \frac{dx}{x^3+1} &= \frac{1}{3}\int\left(\frac{1}{x+1} - \frac{1}{2}\cdot\frac{(2x-1)-3}{x^2-x+1}\right)dx \\
&= \frac{1}{6}\int\left(\frac{2}{x+1} - \frac{2x-1}{x^2-x+1} + \frac{3}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)dx \\
&= \frac{1}{6}\left\{2\log|x+1| - \log(x^2-x+1) + 3\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right\} \\
&= \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.
\end{aligned}$$

(4) 部分分数に分解すると，

$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} - \frac{x-1}{x^2-1} = \frac{1}{x} - \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1}.$$

よって

$$\begin{aligned}
\int \frac{x+1}{x(x^2+1)}\,dx &= \int\left(\frac{1}{x} - \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx \\
&= \log|x| - \frac{1}{2}\log(x^2+1) + \tan^{-1}x \\
&= \log\left|\frac{x}{\sqrt{x^2+1}}\right| + \tan^{-1}x.
\end{aligned}$$

(5) $\dfrac{x^3}{(x-1)(x-2)} = x+3+\dfrac{7x-6}{(x-1)(x-2)} = x+3-\dfrac{1}{x-1}+\dfrac{8}{x-2}$ なので，

$$\begin{aligned}
\int \frac{x^3}{(x-1)(x-2)}\,dx &= \int\left(x+3-\frac{1}{x-1}+\frac{8}{x-2}\right)dx \\
&= \frac{x^2}{2} + 3x - \log|x-1| + 8\log|x-2|.
\end{aligned}$$

(6) $e^x = X$ とおいて，X に関して部分分数に分解すると，

$$\frac{e^x}{e^{2x}-1} = \frac{X}{X^2-1} = \frac{1}{2}\left(\frac{X}{X-1} - \frac{X}{X+1}\right) = \frac{1}{2}\left(\frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}\right).$$

よって

$$\int \frac{e^x}{e^{2x}-1}\,dx = \frac{1}{2}\int\left(\frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}\right)dx = \frac{1}{2}\left(\log|e^x-1| - \log|e^x+1|\right)$$

$$= \frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right| = \frac{1}{2}\log\left|\tanh\frac{x}{2}\right|.$$

問 9. (1)　$a>0$ のとき，$\sqrt{ax^2+bx+c}=t-\sqrt{a}x$ とおくと，

$$x=\frac{t^2-c}{2\sqrt{a}t+b},\quad dx=\frac{2\left(\sqrt{a}t^2+bt+\sqrt{a}c\right)}{(2\sqrt{a}t+b)^2}\,dt,\quad \sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2+bt+\sqrt{a}c}{2\sqrt{a}t+b}.$$

よって

$$I=\int R\left(\frac{t^2-c}{2\sqrt{a}t+b},\frac{\sqrt{a}t^2+bt+\sqrt{a}c}{2\sqrt{a}t+b}\right)\cdot\frac{2\left(\sqrt{a}t^2+bt+\sqrt{a}c\right)}{(2\sqrt{a}t+b)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

(2)　$a<0$ のとき，$\sqrt{\dfrac{x-\alpha}{\beta-x}}=t$ とおくと，

$$x=\frac{\beta t^2+\alpha}{t^2+1},\quad dx=\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となる．また，解と係数の関係

$$\alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a}$$

を用いれば，

$$ax^2+bx+c=\frac{\{2a\alpha\beta+b(\alpha+\beta)+2c\}t^2}{(t^2+1)^2}=\frac{(b^2-4ac)t^2}{-a(t^2+1)^2}$$

$$\therefore\quad \sqrt{ax^2+bx+c}=\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}$$

を得る．よって

$$I=\int R\left(\frac{\beta t^2+\alpha}{t^2+1},\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}\right)\cdot\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

一方，$a<0$ のとき，$\sqrt{\dfrac{\beta-x}{x-\alpha}}=t$ とおくと，

$$x=\frac{\alpha t^2+\beta}{t^2+1},\quad dx=-\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となる．また，

$$ax^2+bx+c=\frac{\{2a\alpha\beta+b(\alpha+\beta)+2c\}t^2}{(t^2+1)^2}=\frac{(b^2-4ac)t^2}{-a(t^2+1)^2}$$

$$\therefore\quad \sqrt{ax^2+bx+c}=\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}.$$

よって

$$I=-\int R\left(\frac{\alpha t^2+\beta}{t^2+1},\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}\right)\cdot\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

参考．$a>0$ のとき，$\sqrt{ax^2+bx+c}=t+\sqrt{a}x$ とおくと，

$$x=\frac{-t^2+c}{2\sqrt{a}t-b},\quad dx=-\frac{2\left(\sqrt{a}t^2-bt+\sqrt{a}c\right)}{(2\sqrt{a}t-b)^2}\,dt,\quad \sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2-bt+\sqrt{a}c}{2\sqrt{a}t-b}$$

より

$$I=-\int R\left(\frac{-t^2+c}{2\sqrt{a}t-b},\frac{\sqrt{a}t^2-bt+\sqrt{a}c}{2\sqrt{a}t-b}\right)\cdot\frac{2\left(\sqrt{a}t^2-bt+\sqrt{a}c\right)}{(2\sqrt{a}t-b)^2}\,dt$$

となり，やはり t に関する有理関数の積分に帰着できる．

問 10. (1) $\sqrt{1-x}=t$ とおくと，$x=1-t^2$，$dx=-2t\,dt$．よって

$$\begin{aligned}\int\frac{dx}{x\sqrt{1-x}}&=\int\frac{-2t}{t(1-t^2)}\,dt=\int\frac{2}{t^2-1}\,dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\\&=\log|t-1|-\log|t+1|=\log\left|\frac{t-1}{t+1}\right|\\&=\log\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right|.\end{aligned}$$

(2) $\sqrt{\dfrac{x-1}{x+1}}=t$ とおくと，$x=-\dfrac{t^2+1}{t^2-1}$，$dx=\dfrac{4t}{(t^2-1)^2}\,dt$．よって

$$\int\sqrt{\frac{x-1}{x+1}}\,dx=\int t\cdot\frac{4t}{(t^2-1)^2}\,dt=4\int\frac{(t^2-1)+1}{(t^2-1)^2}\,dt=4\int\left\{\frac{1}{t^2-1}+\frac{1}{(t^2-1)^2}\right\}dt$$

ここで

$$\begin{aligned}\int\frac{dt}{t^2-1}&=\int t'\cdot\frac{1}{t^2-1}\,dt=\frac{t}{t^2-1}+\int\frac{2t^2}{(t^2-1)^2}\,dt\\&=\frac{t}{t^2-1}+2\int\frac{(t^2-1)+1}{(t^2-1)^2}\,dt\\&=\frac{t}{t^2-1}+2\int\frac{dt}{t^2-1}+2\int\frac{dt}{(t^2-1)^2}\end{aligned}$$

より

$$\int\frac{dt}{(t^2-1)^2}=-\frac{1}{2}\left(\frac{t}{t^2-1}+\int\frac{dt}{t^2-1}\right).$$

よって

$$\begin{aligned}\int\sqrt{\frac{x-1}{x+1}}\,dx&=4\int\frac{dt}{t^2-1}-2\left(\frac{t}{t^2-1}+\int\frac{dt}{t^2-1}\right)\\&=-\frac{2t}{t^2-1}+2\int\frac{dt}{t^2-1}=-\frac{2t}{t^2-1}+\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\\&=-\frac{2t}{t^2-1}+\log\left|\frac{t-1}{t+1}\right|=-\frac{2\sqrt{\frac{x-1}{x+1}}}{\frac{x-1}{x+1}-1}+\log\left|\frac{\sqrt{\frac{x-1}{x+1}}-1}{\sqrt{\frac{x-1}{x+1}}+1}\right|\\&=\sqrt{x^2-1}+\log\left|\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\right|\\&=\sqrt{x^2-1}+\log\left|\frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}{x+1-(x-1)}\right|\\&=\sqrt{x^2-1}+\log\left|x-\sqrt{x^2-1}\right|.\end{aligned}$$

(3) $\sqrt{x^2+x+1}=t-x$ とおくと，

$$x=\frac{t^2-1}{2t+1},\quad dx=\frac{2(t^2+t+1)}{(2t+1)^2}\,dt,\quad \sqrt{x^2+x+1}=\frac{t^2+t+1}{2t+1}.$$

よって

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{2t+1}{t^2-1}\cdot\frac{2t+1}{t^2+t+1}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt$$
$$= \int \frac{2}{(t-1)(t+1)}\,dt = \int \left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$
$$= \log\left|\frac{t-1}{t+1}\right| = \log\left|\frac{\sqrt{x^2+x+1}+x-1}{\sqrt{x^2+x+1}+x+1}\right|.$$

参考. $\sqrt{x^2+x+1} = t+x$ とおくと,

$$x = -\frac{t^2-1}{2t-1},\quad dx = -\frac{2(t^2-t+1)}{(2t-1)^2}\,dt,\quad \sqrt{x^2+x+1} = \frac{t^2-t+1}{2t-1}.$$

よって

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{2t-1}{t^2-1}\cdot\frac{2t-1}{t^2-t+1}\cdot\frac{2(t^2-t+1)}{(2t-1)^2}\,dt$$
$$= \int \frac{2}{t^2-1}\,dt = \log\left|\frac{t-1}{t+1}\right|$$
$$= \log\left|\frac{\sqrt{x^2+x+1}-x-1}{\sqrt{x^2+x+1}-x+1}\right|.$$

(4) $\sqrt{4x-3-x^2} = \sqrt{(x-1)(3-x)} = (3-x)\sqrt{\dfrac{x-1}{3-x}}$ より

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{dx}{x(3-x)\sqrt{\dfrac{x-1}{3-x}}}.$$

そこで, $\sqrt{\dfrac{x-1}{3-x}} = t$ とおくと, $x = \dfrac{3t^2+1}{t^2+1}$, $dx = \dfrac{4t}{(t^2+1)^2}\,dt$, $3-x = \dfrac{2}{t^2+1}$. よって

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{t^2+1}{3t^2+1}\cdot\frac{t^2+1}{2}\cdot\frac{1}{t}\,\frac{4t}{(t^2+1)^2}\,dt = 2\int \frac{dt}{3t^2+1}$$
$$= \frac{2}{3}\int \frac{dt}{t^2+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{3}\cdot\sqrt{3}\tan^{-1}\sqrt{3}t$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3(x-1)}{3-x}}.$$

参考. $\sqrt{4x-3-x^2} = \sqrt{(x-1)(3-x)} = (x-1)\sqrt{\dfrac{3-x}{x-1}}$ より

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{dx}{x(x-1)\sqrt{\dfrac{3-x}{x-1}}}.$$

そこで, $\sqrt{\dfrac{3-x}{x-1}} = t$ とおくと, $x = \dfrac{t^2+3}{t^2+1}$, $dx = -\dfrac{4t}{(t^2+1)^2}\,dt$, $x-1 = \dfrac{2}{t^2+1}$. よって

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{t^2+1}{t^2+3}\cdot\frac{t^2+1}{2}\cdot\frac{1}{t}\,\frac{-4t}{(t^2+1)^2}\,dt = -2\int \frac{dt}{t^2+3}$$

$$= -\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} = -\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3-x}{3(x-1)}}.$$

参考.　$x > 0$ のとき，$\tan^{-1}x + \tan^{-1}\dfrac{1}{x} = \dfrac{\pi}{2}$ である．

(5)　$\tan\dfrac{x}{2} = t$ とおくと，$dx = \dfrac{2}{t^2+1}\,dt$，$\cos x = \dfrac{1-t^2}{1+t^2}$．よって

$$\int \frac{dx}{\cos x} = \int \frac{1+t^2}{1-t^2}\cdot\frac{2}{t^2+1}\,dt = \int \frac{2}{1-t^2}\,dt = \int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$$
$$= \log\left|\frac{1+t}{1-t}\right| = \log\left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}-1}\right| = \log\left|\frac{1+\sin x}{\cos x}\right|.$$

参考.
$$\int\frac{dx}{\cos x} = \int\frac{\cos x}{\cos^2 x}\,dx = \int\frac{\cos x}{1-\sin^2 x}\,dx = \int\frac{\cos x}{(1-\sin x)(1+\sin x)}\,dx$$
$$= \frac{1}{2}\int\left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)dx$$
$$= \frac{1}{2}\int\left\{-\frac{(1-\sin x)'}{1-\sin x}+\frac{(1+\sin x)'}{1+\sin x}\right\}dx$$
$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

(6)　$\tan\dfrac{x}{2} = t$ とおくと，$dx = \dfrac{2}{t^2+1}\,dt$，$\sin x = \dfrac{2}{t^2+1}$．よって

$$\int\frac{\sin x}{1+\sin x}\,dx = \int\left(1-\frac{1}{1+\sin x}\right)dx = x - \int\frac{dx}{1+\sin x}$$
$$= x - \int\frac{t^2+1}{(t+1)^2}\cdot\frac{2}{t^2+1}\,dt = x - 2\int\frac{dt}{(t+1)^2}$$
$$= x + \frac{2}{t+1} = x + \frac{2}{\tan\frac{x}{2}+1}.$$

参考.
$$\int\frac{\sin x}{1+\sin x}\,dx = \int\frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx = \int\frac{\sin x-\sin^2 x}{1-\sin^2 x}\,dx$$
$$= \int\frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\,dx = \int\left(\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}+1\right)dx$$
$$= \frac{1}{\cos x} - \tan x + x = x + \frac{1-\sin x}{\cos x}.$$

問 11.　(1)　$e^x = t$ とおくと，$dx = \frac{dt}{t}$．よって

$$\int\frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx = \int\frac{e^{2x}-1}{e^{2x}+1}\,dx = \int\frac{t^2-1}{t(t^2+1)}\,dt = \int\left(\frac{2t}{t^2+1}-\frac{1}{t}\right)dt$$
$$= \log(t^2+1) - \log|t| = \log\frac{t^2+1}{t}$$
$$= \log(e^x+e^{-x}).$$

参考.
$$\int\frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx = \int\frac{(e^x+e^{-x})'}{e^x+e^{-x}}\,dx = \log(e^x+e^{-x})$$

参考. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \tanh x\,dx = \log\cosh x = \log\frac{e^x + e^{-x}}{2} = \log\left(e^e + e^{-x}\right) - \log 2.$

(2) $\log x = t$ とおくと，$\dfrac{dx}{x} = dt$．よって

$$\int \frac{(\log x + 3)^5}{x}\,dx = \int (t+3)^5\,dt = \frac{1}{6}(t+3)^6 = \frac{1}{6}(\log x + 3)^6.$$

参考. $\displaystyle\int \frac{(\log x + 3)^5}{x}\,dx = \int (\log x + 3)'(\log x + 3)^5\,dx = \frac{1}{6}(\log x + 3)^6$

(3) $\sin x = t\ \left(-\dfrac{\pi}{2} \leqq x \leqq \dfrac{\pi}{2}\right)$ とおくと，$\cos x\,dx = dt$．よって

$$\int \frac{\cos x}{1 + \sin^2 x}\,dx = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1}(\sin x).$$

(4) $\cos x = t\ \ (0 \leqq x \leqq \pi)$ とおくと，$\sin x\,dx = -dt$．よって

$$\begin{aligned}
\int \frac{\cos^2 x \sin x}{3\cos^2 x + \sin^2 x}\,dx &= \int \frac{\cos^2 x \sin x}{1 + 2\cos^2 x}\,dx = \int \frac{-t^2}{1+2t^2}\,dt \\
&= -\frac{1}{2}\int\left(1 - \frac{\frac{1}{2}}{t^2 + \frac{1}{2}}\right)dt \\
&= -\frac{t}{2} + \frac{1}{4}\cdot\sqrt{2}\tan^{-1}\left(\sqrt{2}t\right) \\
&= -\frac{1}{2}\cos x + \frac{\sqrt{2}}{4}\tan^{-1}\left(\sqrt{2}\cos x\right).
\end{aligned}$$

(5) $\tan x = t\ \left(-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right)$ とおくと，$dx = \dfrac{dt}{1+t^2}$．よって

$$\begin{aligned}
\int \frac{dx}{1+\tan x} &= \int \frac{1}{1+t}\cdot\frac{1}{1+t^2}\,dt = \frac{1}{2}\int\left(\frac{1}{1+t} + \frac{1-t}{1+t^2}\right)dt \\
&= \frac{1}{2}\int\left(\frac{1}{1+t} - \frac{t}{1+t^2} + \frac{1}{1+t^2}\right)dt \\
&= \frac{1}{2}\left\{\log|1+t| - \frac{1}{2}\log(1+t^2) + \tan^{-1} t\right\} \\
&= \frac{1}{4}\log\frac{(1+\tan x)^2}{1+\tan^2 x} + \frac{x}{2} = \frac{1}{4}\log(\sin x + \cos x)^2 + \frac{x}{2} \\
&= \frac{1}{4}\log(1 + \sin 2x) + \frac{x}{2}.
\end{aligned}$$

(6) $\tan x = t\ \left(-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right)$ とおくと，$dx = \dfrac{dt}{1+t^2}$．よって

$$\begin{aligned}
\int \frac{\tan^2 x}{3 + \tan^3 x}\,dx &= \int \frac{t^2}{3+t^2}\cdot\frac{1}{1+t^2}\,dt = \frac{1}{2}\int\left(\frac{3}{3+t^2} - \frac{1}{1+t^2}\right)dt \\
&= \frac{3}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} - \frac{1}{2}\tan^{-1} t \\
&= \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right) - \frac{x}{2}.
\end{aligned}$$

3.2 定積分

問 1. $f(x)=x^2$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n-1)$$
$$= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{1}{3}.$$

問 2. $g(x)$ が恒等的に 0 のときは，任意の $c\in(a,b)$ に対して成り立つ．よって，少なくとも 1 点で $g(x)>0$ とする．このとき，定理 6 の (4) より，$\displaystyle\int_a^b g(x)\,dx>0$ となる．

さて，定理 1.15 (最大値・最小値の定理) より，$f(x)$ の $[a,b]$ における最小値を m，最大値を M とする．$m=M$ のときは，$f(x)$ は定数関数で，$f(x)=m$ となる．よって，任意の $c\in(a,b)$ に対して

$$\int_a^b f(x)g(x)\,dx = m\int_a^b g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

となり成り立つ．そこで，以下では $m<M$ と仮定する．このとき，$m\leqq f(x)\leqq M$ かつ $g(x)\geqq 0$ より，$mg(x)\leqq f(x)g(x)\leqq Mg(x)$ なので，

$$m\int_a^b g(x)\,dx \leqq \int_a^b f(x)g(x)\,dx \leqq M\int_a^b g(x)\,dx.$$

そこで

$$\lambda := \frac{\displaystyle\int_a^b f(x)g(x)\,dx}{\displaystyle\int_a^b g(x)\,dx}$$

とおくと，$m\leqq\lambda\leqq M$ である．

- $\lambda=m$ のときは，$f(x)$ が最小となる x の値を $c\in[a,b]$ とすれば，$\lambda=m=f(c)$.
- $\lambda=M$ のときは，$f(x)$ が最大となる x の値を $c\in[a,b]$ とすれば，$\lambda=M=f(x)$.
- $m<\lambda<M$ のときは，定理 1.14 (中間値の定理) より，$\lambda=f(c)$ となる $c\in(a,b)$ が存在する．

よって，いずれの場合でも，$\lambda=f(c)$ となる $c\in[a,b]$ が存在するので，

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

が成り立つ．

参考． $g(x)=1$ のときは積分の平均値の定理 (定理 3.7) である．

問 3. (1) $\displaystyle\int_0^1 (2x-1)^8\,dx = \left[\frac{1}{18}(2x-1)^9\right]_0^1 = \frac{2}{18} = \frac{1}{9}$

(2) $\displaystyle\int_0^1 x\sqrt[3]{x}\,dx = \int_0^1 x^{\frac{4}{3}}\,dx = \left[\frac{3}{7}x^{\frac{7}{3}}\right]_0^1 = \frac{3}{7}$

(3) $\displaystyle\int_0^1 2^x\,dx = \left[\frac{2^x}{\log 2}\right]_0^1 = \frac{1}{\log 2}(2-1) = \frac{1}{\log 2}$

問 4. (1) $m > 1$ は任意の定数とし，

$$F(x) = \int_1^x \frac{dt}{\sqrt{t^2+1}} \quad (x \in \mathbb{R})$$

とおく．関数 $f(t) = \dfrac{1}{t^2+1}$ は $\mathbb{R}$ で連続なので，定理 3.8 (微分積分学の基本定理) より，

$$G(x) = \int_{-m}^x f(t)dt \quad (-m \leqq x \leqq m)$$

は $[-m, m]$ で微分可能で，

$$G'(x) = f(x) = \frac{1}{\sqrt{x^2+1}}$$

となる．ここで，

$$F(x) = \int_1^x f(t)dt = \int_1^{-m} f(t)dt + \int_{-m}^x f(t)dt = \int_1^{-m} f(t)dt + G(x)$$

なので，$F(x)$ は $[-m, m]$ で微分可能で，$F'(x) = G'(x) = \dfrac{1}{\sqrt{x^2+1}}$ となる．$m > 1$ は任意なので，$F(x)$ は $\mathbb{R}$ で微分可能で，$F'(x) = \dfrac{1}{\sqrt{x^2+1}}$ が成り立つ．

(2) $m > 0$ は任意の定数とし，

$$F(x) = \int_0^x e^t \cos t dt \quad (x \in \mathbb{R})$$

とおく．関数 $f(t) = e^t \cos t$ は $\mathbb{R}$ で連続なので，微分積分学の基本定理より，

$$G(x) = \int_{-m}^x f(t)dt \quad (-m \leqq x \leqq m)$$

は $[-m, m]$ で微分可能で，

$$G'(x) = f(x) = e^x \cos x$$

となる．ここで，

$$F(x) = \int_0^x f(t)dt = \int_0^{-m} f(t)dt + \int_{-m}^x f(t)dt = \int_0^{-m} f(t)dt + G(x)$$

なので，$F(x)$ は $[-m, m]$ で微分可能で，$F'(x) = G'(x) = e^x \cos x$ となる．$m > 0$ は任意なので，$F(x)$ は $\mathbb{R}$ で微分可能で，$F'(x) = e^x \cos x$ が成り立つ．さて，与式を変形すると

$$\int_x^{x^2} e^t \cos t\, dt = \int_0^{x^2} e^t \cos t\, dt - \int_0^x e^t \cos t\, dt = F(x^2) - F(x).$$

である．よって，合成関数の微分法より，上式の左辺は $\mathbb{R}$ で微分可能で，

$$\begin{aligned}\frac{d}{dx}\int_x^{x^2} e^t \cos t\, dt &= \frac{d}{dx}\left\{F(x^2) - F(x)\right\} = 2xF'(x^2) - F'(x) \\ &= 2xf(x^2) - f(x) = 2xe^{x^2}\cos x^2 - e^x \cos x.\end{aligned}$$

(3) (1) や (2) と同様にして，微分積分学の基本定理により，

$$F(x) = \int_1^x te^t\, dt, \quad G(x) = \int_1^x e^t\, dt$$

はともに $\mathbb{R}$ で微分可能で，$F'(x) = xe^x$，$G'(x) = e^x$ となる．さて，与式を変形すると

$$\int_1^{x^2+1} (t-x)e^t\,dt = \int_1^{x^2+1} te^t\,dt - x\int_1^{x^2+1} e^t\,dt = F(x^2+1) - xG(x^2+1)$$

である．よって，合成関数の微分法より，上式の左辺は $\mathbb{R}$ で微分可能で，

$$\begin{aligned}
\frac{d}{dx}\int_1^{x^2+1}(t-x)e^t\,dt &= \frac{d}{dx}\left\{F(x^2+1) - xG(x^2+1)\right\} \\
&= 2xF'(x^2+1) - G(x^2+1) - 2x^2G'(x^2+1) \\
&= 2x(x^2+1)e^{x^2+1} - \int_1^{x^2+1} e^t\,dt - 2x^2e^{x^2+1} \\
&= 2x(x^2+1)e^{x^2+1} - \Big[e^t\Big]_1^{x^2+1} - 2x^2e^{x^2+1} \\
&= 2x(x^2+1)e^{x^2+1} - e^{x^2+1} + e - 2x^2e^{x^2+1} \\
&= (2x^3 - 2x^2 + 2x - 1)e^{x^2+1} + e.
\end{aligned}$$

問 5.　(1)

$$\begin{aligned}
\int_1^2 x(x-1)^7\,dx &= \left[\frac{x}{8}(x-1)^8\right]_1^2 - \frac{1}{8}\int_1^2 (x-1)^8\,dx \\
&= \frac{1}{4} - \frac{1}{8}\left[\frac{1}{9}(x-1)^9\right]_1^2 = \frac{17}{72}.
\end{aligned}$$

(2)　$\log x = t$ とおくと，$\dfrac{dx}{x} = dt$．また，x が 1 から e まで動くとき，t は 0 から 1 まで動く．よって

$$\int_1^e \frac{(\log x)^2}{x}\,dx = \int_0^1 t^2\,dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}.$$

(3)　$\sqrt{2-x} = t$ とおくと，$x = 2 - t^2$，$dx = -2t\,dt$．また，x が 0 から 1 まで動くとき，t は $\sqrt{2}$ から 1 まで動く．よって

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{2-x}}\,dx &= \int_{\sqrt{2}}^1 \frac{2-t^2}{t}\cdot(-2t\,dt) = 2\int_{\sqrt{2}}^1 (t^2-2)\,dt = 2\left[\frac{1}{3}t^3 - 2t\right]_{\sqrt{2}}^1 \\
&= \frac{2}{3}(1-2\sqrt{2}) - 4(1-\sqrt{2}) = \frac{2}{3}(4\sqrt{2}-5).
\end{aligned}$$

(4)　$\tan\dfrac{x}{2} = t$ とおくと，$dx = \dfrac{2\,dt}{t^2+1}$，$\sin x = \dfrac{2t}{t^2+1}$．また，$x$ が 0 から $\dfrac{\pi}{2}$ まで動くとき，t は 0 から 1 まで動く．よって

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{dx}{2+3\sin x} &= \int_0^1 \frac{1}{2+3\cdot\frac{2t}{t^2+1}}\cdot\frac{2\,dt}{t^2+1} = \int_0^1 \frac{dt}{t^2+3t+1} \\
&= \int_0^1 \frac{dt}{\left(t+\frac{3}{2}\right)^2 - \frac{5}{4}} = \left[\frac{1}{\sqrt{5}}\log\left|\frac{t+\frac{3}{2}-\frac{\sqrt{5}}{2}}{t+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|\right]_0^1 \\
&= \frac{1}{\sqrt{5}}\left\{\log\left|\frac{\frac{5}{2}-\frac{\sqrt{5}}{2}}{\frac{5}{2}+\frac{\sqrt{5}}{2}}\right| - \log\left|\frac{\frac{3}{2}-\frac{\sqrt{5}}{2}}{\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|\right\} \\
&= \frac{1}{\sqrt{5}}\log\left|\frac{(5-\sqrt{5})(3+\sqrt{5})}{(5+\sqrt{5})(3-\sqrt{5})}\right| = \frac{1}{\sqrt{5}}\log\frac{5+\sqrt{5}}{5-\sqrt{5}}.
\end{aligned}$$

(5)　$\sqrt{1+e^x}=t$ とおくと，$1+e^x=t^2$，$e^x dx=2t\,dt$．また，x が 0 から 1 まで動くとき，t は $\sqrt{2}$ から $\sqrt{1+e}$ まで動く．よって

$$\begin{aligned}\int_0^1 e^{2x}\sqrt{1+e^x}\,dx &= \int_0^1 e^x\sqrt{1+e^x}\cdot e^x dx = \int_{\sqrt{2}}^{\sqrt{1+e}}(t^2-1)\cdot t\cdot(2t\,dt)\\ &= 2\int_{\sqrt{2}}\sqrt{1+e}t^2(t^2-1)\,dt = 2\int_{\sqrt{2}}\sqrt{1+e}(t^4-t^2)\,dt\\ &= 2\left[\frac{1}{5}t^5-\frac{1}{3}t^3\right]_{\sqrt{2}}^{\sqrt{1+e}}\\ &= \frac{2}{5}\left\{(1+e)^2\sqrt{1+e}-4\sqrt{2}\right\}-\frac{2}{3}\left\{(1+e)\sqrt{1+e}-2\sqrt{2}\right\}\\ &= \frac{2}{15}(3e-2)(1+e)\sqrt{1+e}-\frac{4}{15}\sqrt{2}.\end{aligned}$$

(6)
$$\begin{aligned}\int_1^3 x\,(\log x)^2\,dx &= \left[\frac{x^2}{2}(\log x)^2\right]_1^2-\int_1^3\frac{x^2}{2}\cdot\frac{2\log x}{x}\,dx = \frac{9}{2}(\log 3)^2-\int_1^3 x\log x\,dx\\ &= \frac{9}{2}(\log 3)^2-\left\{\left[\frac{x^2}{2}\log x\right]_1^3-\int_1^3\frac{x^2}{2}\cdot\frac{1}{x}\,dx\right\}\\ &= \frac{9}{2}(\log 3)^2-\frac{9}{2}\log 3+\frac{1}{2}\int_1^3 x\,dx\\ &= \frac{9}{2}(\log 3)^2-\frac{9}{2}\log 3+\frac{1}{2}\left[\frac{x^2}{2}\right]_1^3\\ &= \frac{9}{2}(\log 3)^2-\frac{9}{2}\log 3+2.\end{aligned}$$

(7)
$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2\cos x\,dx &= \Big[x^2\sin x\Big]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\sin x\,dx = \frac{\pi^2}{4}-2\int_0^{\frac{\pi}{2}}x\sin x\,dx\\ &= \frac{\pi^2}{4}-2\left\{\Big[x(-\cos x)\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx\right\} = \frac{\pi^2}{4}-2\int_0^{\frac{\pi}{2}}\cos x\,dx\\ &= \frac{\pi^2}{4}-2\Big[\sin x\Big]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}-2.\end{aligned}$$

(8)
$$\begin{aligned}\int_0^1 x^2e^{-x}dx &= \Big[x^2(-e^{-x}\Big]_0^1+2\int_0^1 xe^{-x}dx = -e^{-1}+2\int_0^1 xe^{-x}dx\\ &= -e^{-1}+2\left\{\Big[x(-e^{-x})\Big]_0^1+\int_0^1 e^{-x}dx\right\} = -e^{-1}+2\left\{-e^{-1}+\int_0^1 e^{-x}dx\right\}\\ &= -3e^{-1}+2\Big[-e^{-x}\Big]_0^1 = 2-5e^{-1}.\end{aligned}$$

(9)
$$\begin{aligned}\int_0^1\cos^{-1}x\,dx &= \Big[x\cos^{-1}x\Big]_0^1+\int_0^1\frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^1\frac{-2x}{\sqrt{1-x^2}}\,dx\\ &= -\frac{1}{2}\Big[2\sqrt{1-x^2}\Big]_0^1 = 1.\end{aligned}$$

問 6.　$n \geqq 3$ とする．部分積分公式を用いて計算すると，

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\frac{1}{\cos^2 x} - 1\right) dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan^{n-2} x}{\cos^2 x}\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^{n-2}}{\cos^2 x}\,dx - I_{n-2}.$$

ここで，$\tan x = t$ とおくと，$\dfrac{dx}{\cos^2 x} = dt$．また，$x$ が 0 から $\dfrac{\pi}{4}$ まで動くとき，t は 0 から 1 まで動く．よって

$$\int_0^{\frac{\pi}{4}} \frac{\tan^{n-2} x}{\cos^2 x}\,dx = \int_0^1 t^{n-2}\,dt = \left[\frac{t^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}.$$

ゆえに

$$I_n = \frac{1}{n-1} - I_{n-2} \quad (n \geqq 3)$$

が成り立つ．

上で示した漸化式より，

$$I_3 = \frac{1}{2} - I_1, \quad I_4 = \frac{1}{3} - I_2, \quad I_5 = \frac{1}{4} - I_3.$$

よって

$$I_6 = \frac{1}{5} - I_4 = \frac{1}{5} - \left(\frac{1}{3} - I_2\right) = -\frac{2}{15} + I_2 \qquad ①$$

$$I_7 = \frac{1}{6} - I_5 = \frac{1}{6} - \left(\frac{1}{4} - I_3\right) = -\frac{1}{12} + I_3 = -\frac{1}{12} + \left(\frac{1}{2} - I_1\right) = \frac{5}{12} - I_1. \qquad ②$$

ここで

$$I_1 = \int_0^{\frac{\pi}{4}} \tan x\,dx = \Big[-\log|\cos x|\Big]_0^{\frac{\pi}{4}} = -\log\left|\cos\frac{\pi}{4}\right| = -\log\frac{1}{\sqrt{2}} = \frac{1}{2}\log 2. \qquad ③$$

また，$\tan x = t$ とおくと，$dx = \dfrac{dt}{t^2+1}$．さらに，x が 0 から $\dfrac{\pi}{4}$ まで動くとき，t は 0 から 1 まで動く．よって

$$I_2 = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^1 \frac{t^2}{t^2+1}\,dt = \int_0^1 \left(1 - \frac{1}{t^2+1}\right) dt = \Big[t - \tan^{-1} t\Big]_0^1$$
$$= 1 - (\tan^{-1} 1 - \tan^{-1} 0) = 1 - \frac{\pi}{4}. \qquad ④$$

よって，①，②，③，④より，

$$I_6 = \frac{13}{15} - \frac{\pi}{4}, \quad I_7 = \frac{5}{12} - \frac{1}{2}\log 2.$$

問 7.　(1)　$f(x) = \sqrt{1-x^2}$ は $[0,1]$ で連続．よって，区分求積公式より，

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{\sqrt{n^2-i^2}}{n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \sqrt{1 - \left(\frac{i}{n}\right)^2} = \int_0^1 \sqrt{1-x^2}\,dx$$
$$= \left[\frac{1}{2}\left(\sin^{-1} + x\sqrt{1-x^2}\right)\right]_0^1 = \frac{1}{2}\left(\sin^{-1} 1 - \sin^{-1} 0\right) = \frac{\pi}{4}.$$

(2)　$f(x) = x\sin\pi x$ は $[0,1]$ で連続．よって，区分求積公式より，

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\sin\frac{i\pi}{n} = \int_0^1 x\sin\pi x\,dx = \left[x\left(-\frac{1}{\pi}\cos\pi x\right)\right]_0^1 + \frac{1}{\pi}\int_0^1\cos\pi x\,dx$$
$$= \frac{1}{\pi} + \frac{1}{\pi}\left[\frac{1}{\pi}\sin\pi x\right]_0^1 = \frac{1}{\pi}.$$

(3)　$f(x) = \sqrt{x}$ は $[0,1]$ で連続．よって，区分求積公式より，

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\sqrt{i} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}} = \int_0^1\sqrt{x}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}.$$

(4)　$f(x) = \dfrac{1}{4-x^2}$ は $[0,1]$ で連続．よって，区分求積公式より，

$$\lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{4n^2-i^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{n^2}{4n^2-i^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{4-\left(\frac{i}{n}\right)^2} = \int_0^1\frac{dx}{4-x^2}$$
$$= \frac{1}{4}\int_0^1\left(\frac{1}{2-x}+\frac{1}{2+x}\right)dx = \frac{1}{4}\left[\log\left|\frac{2+x}{2-x}\right|\right]_0^1$$
$$= \frac{1}{4}\big(\log 3 - \log 1\big) = \frac{1}{4}\log 3.$$

3.3　広義積分

問 1.　(1)　$\dfrac{1}{\sqrt[3]{x}}$ は $(0,2]$ で連続．よって

$$\int_0^2 \frac{dx}{\sqrt[3]{x}} = \lim_{\alpha\to+0}\int_\alpha^2 \frac{dx}{\sqrt[3]{x}} = \lim_{\alpha\to+0}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_\alpha^2 = \lim_{\alpha\to+0}\frac{3}{2}\left(2^{\frac{2}{3}} - \alpha^{\frac{2}{3}}\right) = \frac{3}{2}\sqrt[3]{4}.$$

(2)　$\dfrac{1}{\sqrt{2-x}}$ は $[0,2)$ で連続．よって

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{2-x}} &= \lim_{\beta\to2-0}\int_0^\beta \frac{dx}{\sqrt{2-x}} = \lim_{\beta\to2-0}\left[-2\sqrt{2-x}\right]_0^\beta \\ &= \lim_{\beta\to2-0} 2\left(\sqrt{2} - \sqrt{2-\beta}\right) = 2\sqrt{2}.\end{aligned}$$

(3)　$\dfrac{\log x}{x}$ は $(0,1]$ で連続．よって

$$\begin{aligned}\int_0^1 \frac{\log x}{x}\,dx &= \lim_{\alpha\to+0}\int_\alpha^1 \frac{\log x}{x}\,dx = \lim_{\alpha\to+0}\left[\frac{1}{2}(\log x)^2\right]_\alpha^1 \\ &= -\lim_{\alpha\to+0}\frac{1}{2}(\log\alpha)^2 = -\infty \quad (\text{発散}).\end{aligned}$$

(4)　$\dfrac{1}{\sqrt{(x-1)(2-x)}}$ は $(1,2)$ で連続．よって

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta \frac{dx}{\sqrt{(x-1)(2-x)}}\,dx\right) \\ &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta \frac{dx}{\sqrt{\frac{1}{4} - \left(x-\frac{3}{2}\right)^2}}\,dx\right) \\ &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\left[\sin^{-1}(2x-3)\right]_\alpha^\beta\right) \\ &= \lim_{\beta\to2-0}\sin^{-1}(2\beta-3) - \lim_{\alpha\to1+0}\sin^{-1}(2\alpha-3) \\ &= \sin^{-1}1 - \sin^{-1}(-1) = \pi.\end{aligned}$$

(5)　$\dfrac{1}{\sin x}$ は $\left(0, \dfrac{\pi}{2}\right]$ で連続．よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x} = \lim_{\alpha\to+0}\int_\alpha^{\frac{\pi}{2}} \frac{dx}{\sin x}.$$

ここで，$\tan\dfrac{x}{2} = t$ とおくと，$dx = \dfrac{2}{1+t^2}dt$，$\sin x = \dfrac{2t}{1+t^2}$ なので，

$$\int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t}\frac{2}{1+t^2}dt = \int \frac{dt}{t} = \log|t| = \log\left|\tan\frac{x}{2}\right|.$$

よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x} = \lim_{\alpha\to+0}\left[\log\left|\tan\frac{x}{2}\right|\right]_\alpha^{\frac{\pi}{2}} = -\lim_{\alpha\to+0}\log|\alpha| = \infty \quad (\text{発散}).$$

別解. $\dfrac{1}{\sin x}$ を変形すると,

$$\frac{1}{\sin x}=\frac{\sin x}{\sin^2 x}=\frac{\sin x}{1-\cos^2 x}=\frac{\sin x}{(1-\cos x)(1+\cos x)}=\frac{1}{2}\left(\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}\right)$$

となる．よって,

$$\begin{aligned}I&=\frac{1}{2}\left\{\int\frac{\sin x}{1-\cos x}\,dx+\int\frac{\sin x}{1+\cos x}\,dx\right\}\\&=\frac{1}{2}\left\{\log|1-\cos x|-\log|1+\cos x|\right\}=\frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right|\end{aligned}$$

である．ゆえに,

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sin x}=\lim_{\alpha\to+0}\left[\frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right|\right]_\alpha^{\frac{\pi}{2}}=-\frac{1}{2}\lim_{\alpha\to+0}\log\left|\frac{1-\cos\alpha}{1+\cos\alpha}\right|=\infty\quad(\text{発散}).$$

(6) $\dfrac{1}{\sqrt{|x(x-2)|}}$ は $(0,2)$ と $(2,3]$ で連続．ここで

$$\begin{aligned}\int_0^2\frac{dx}{\sqrt{|x(x-2)|}}&=\int_0^2\frac{dx}{\sqrt{-x(x-2)}}=\lim_{\alpha\to+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta\frac{dx}{\sqrt{2x-x^2}}\right)\\&=\lim_{\alpha\to+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta\frac{dx}{\sqrt{1-(x-1)^2}}\right)\\&=\lim_{\alpha\to+0}\left(\lim_{\beta\to2-0}\left[\sin^{-1}(x-1)\right]_\alpha^\beta\right)\\&=\lim_{\beta\to2-0}\sin^{-1}(\beta-1)-\lim_{\alpha\to+0}\sin^{-1}(\alpha-1)\\&=\sin^{-1}1-\sin^{-1}(-1)=\pi.\end{aligned}$$

$$\begin{aligned}\int_0^2\frac{dx}{\sqrt{|x(x-2)|}}&=\int_2^3\frac{dx}{\sqrt{x(x-2)}}=\lim_{\alpha\to2+0}\int_\alpha^3\frac{dx}{\sqrt{x^2-2x}}\\&=\lim_{\alpha\to2+0}\int_\alpha^3\frac{dx}{\sqrt{(x-1)^2-1}}\\&=\lim_{\alpha\to2+0}\left[\log\left|x-1+\sqrt{x^2-2x}\right|\right]_\alpha^3\\&=\lim_{\alpha\to2+0}\left\{\log\left(2+\sqrt{3}\right)-\log\left(\alpha-1+\sqrt{\alpha^2-2\alpha}\right)\right\}\\&=\log\left(2+\sqrt{3}\right).\end{aligned}$$

よって

$$\int_0^3\frac{dx}{\sqrt{|x(x-2)|}}=\int_0^2\frac{dx}{\sqrt{|x(x-2)|}}+\int_2^3\frac{dx}{\sqrt{|x(x-2)|}}=\pi+\log\left(2+\sqrt{3}\right).$$

問 2. $\dfrac{1}{x^p}$ は $[1,\infty)$ で連続．ゆえに，$p>1$ のとき,

$$\int_1^\infty\frac{dx}{x^p}=\lim_{\beta\to\infty}\int_1^\beta\frac{dx}{x^p}=\lim_{\beta\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^\beta=\lim_{\beta\to\infty}\frac{1}{1-p}\left(\frac{1}{\beta^{p-1}}-1\right)=\frac{1}{p-1}.$$

次に，$p=1$ のとき，

$$\int_1^\infty \frac{dx}{x} = \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x} = \lim_{\beta\to\infty}\Big[\log x\Big]_1^\beta = \lim_{\beta\to\infty}\log\beta = \infty.$$

また，$p<1$ のとき，

$$\int_1^\infty \frac{dx}{x^p} = \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x^p} = \lim_{\beta\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^\beta = \lim_{\beta\to\infty}\frac{1}{1-p}\left(\beta^{1-p}-1\right) = \infty.$$

問 3. (1) $\dfrac{1}{x(1+x^2)}$ は $[1,\infty)$ で連続．よって

$$\begin{aligned}\int_1^\infty \frac{dx}{x(1+x^2)} &= \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x(1+x^2)} = \lim_{\beta\to\infty}\int_1^\beta\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx\\ &= \lim_{\beta\to\infty}\left[\log x - \frac{1}{2}\log\left(1+x^2\right)\right]_1^\beta\\ &= \frac{1}{2}\log 2 + \lim_{\beta\to\infty}\left\{\log\beta - \frac{1}{2}\log\left(1+\beta^2\right)\right\} = \frac{1}{2}\log 2.\end{aligned}$$

(2) $\dfrac{x^3}{1+x^4}$ は $[0,\infty)$ で連続．よって

$$\begin{aligned}\int_0^\infty \frac{x^3}{1+x^4}\,dx &= \lim_{\beta\to\infty}\int_0^\beta \frac{x^3}{1+x^4}\,dx = \lim_{\beta\to\infty}\left[\frac{1}{4}\log\left(1+x^4\right)\right]_0^\beta\\ &= \lim_{\beta\to\infty}\frac{1}{4}\log\left(1+\beta^4\right) = \infty.\end{aligned}$$

(3) xe^{-x^2} は $[0,\infty)$ で連続．よって

$$\int_0^\infty xe^{-x^2}dx = \lim_{\beta\to\infty}\int_0^\beta xe^{-x^2}dx = \lim_{\beta\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^\beta = \lim_{\beta\to\infty}\frac{1}{2}\left(1-e^{-\beta^2}\right) = \frac{1}{2}.$$

(4) $\dfrac{1}{x\sqrt{x^2-1}}$ は $(1,\infty)$ で連続．よって

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{\alpha\to1+0}\left(\lim_{\beta\to\infty}\int_\alpha^\beta \frac{dx}{x\sqrt{x^2-1}}\right).$$

ここで，$\sqrt{x^2-1}=t$ とおくと，$t^2=x^2-1$，$t\,dt = x\,dx$．よって

$$\int\frac{dx}{x\sqrt{x^2-1}} = \int\frac{x}{x^2\sqrt{x^2-1}}\,dx = \int\frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{x^2-1}.$$

ゆえに

$$\begin{aligned}\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to\infty}\Big[\tan^{-1}\sqrt{x^2-1}\Big]_\alpha^\beta\right)\\ &= \lim_{\beta\to\infty}\tan^{-1}\sqrt{\beta^2-1} - \lim_{\alpha\to1+0}\tan^{-1}\sqrt{\alpha^2-1}\\ &= \frac{\pi}{2}.\end{aligned}$$

(5) $\dfrac{\log x}{x^2}$ は $[1,\infty)$ で連続．よって

$$\int_1^\infty \frac{\log x}{x^2}\,dx = \lim_{\beta\to\infty}\int_1^\beta \frac{\log x}{x^2}\,dx.$$

ここで

$$\int \frac{\log x}{x^2}\,dx = \left(-\frac{1}{x}\right)\log x - \int \left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx = -\frac{\log x}{x} + \int \frac{dx}{x^2}$$
$$= -\frac{\log x}{x} - \frac{1}{x} = -\frac{1+\log x}{x}.$$

よって，ロピタルの定理を用いて極限を計算すると，

$$\int_1^\infty \frac{\log x}{x^2}\,dx = \lim_{\beta\to\infty}\left[-\frac{1+\log x}{x}\right]_1^\beta = 1 - \lim_{\beta\to\infty}\frac{1+\log\beta}{\beta} = 1 - \lim_{\beta\to\infty}\frac{1}{\beta} = 1.$$

(6)　$e^{-ax}\sin bx$ は $[0,\infty)$ で連続．よって

$$\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-ax}\sin bx\,dx.$$

ここで

$$\int e^{-ax}\sin bx\,dx = \int\left(-\frac{1}{a}e^{-ax}\right)'\sin bx\,dx$$
$$= -\frac{1}{a}e^{-ax}\sin bx + \frac{b}{a}\int e^{-ax}\cos bx\,dx$$
$$= -\frac{1}{a}e^{-ax}\sin bx - \frac{b}{a^2}e^{-ax}\cos bx - \frac{b^2}{a^2}\int e^{-ax}\sin bx\,dx.$$

よって

$$\int e^{-ax}\sin bx\,dx = -\frac{e^{-ax}}{a^2+b^2}(a\sin bx + b\cos bx).$$

ゆえに

$$\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{\beta\to\infty}\left[-\frac{e^{-ax}}{a^2+b^2}(a\sin bx + b\cos bx)\right]_0^\beta$$
$$= -\lim_{\beta\to\infty}\left\{-\frac{e^{-a\beta}}{a^2+b^2}(a\sin b\beta + b\cos b\beta)\right\} + \frac{b}{a^2+b^2}$$
$$= \frac{b}{a^2+b^2}.$$

問 4.　(1)　e^{-x} は $[0,\infty)$ で連続．よって

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-x}\,dx = \lim_{\beta\to\infty}\Big[-e^{-x}\Big]_0^\beta = \lim_{\beta\to\infty}\left(1-e^{-\beta}\right) = 1.$$

(2)　x^2e^{-x} は $[0,\infty)$ で連続．よって

$$\Gamma(3) = \int_0^\infty x^2e^{-x}\,dx = \lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx.$$

ここで

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$
$$= -(x^2+2x+2)e^{-x}.$$

よって

$$\Gamma(3)=\int_0^\infty x^2e^{-x}\,dx=\lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx=\lim_{\beta\to\infty}\Big[-(x^2+2x+2)e^{-x}\Big]_0^\beta$$
$$=\lim_{\beta\to\infty}\Big\{2-(\beta^2+2\beta+2)e^{-\beta}\Big\}=2.$$

(3) $\dfrac{1}{\sqrt{x(1-x)}}$ は $(0,1)$ で連続．よって

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1\frac{dx}{\sqrt{x(1-x)}}=\lim_{\alpha\to+0}\left(\lim_{\beta\to1-0}\int_\alpha^\beta\frac{dx}{\sqrt{x(1-x)}}\right).$$

ここで

$$\int\frac{dx}{\sqrt{x(1-x)}}=\int\frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}=\sin^{-1}(2x-1).$$

よって

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\lim_{\alpha\to+0}\left(\lim_{\beta\to1-0}\Big[\sin^{-1}(2x-1)\Big]_\alpha^\beta\right)$$
$$=\lim_{\beta\to1-0}\sin^{-1}(2\beta-1)-\lim_{\alpha\to+0}\sin^{-1}(2\alpha-1)$$
$$=\sin^{-1}1-\sin^{-1}(-1)=\pi.$$

(4) $\dfrac{(1-x)^2}{\sqrt{x}}$ は $(0,1]$ で連続．よって

$$B\left(\frac{1}{2},3\right)=\int_0^1\frac{(1-x)^2}{\sqrt{x}}\,dx=\lim_{\alpha\to+0}\int_\alpha^1\frac{(1-x)^2}{\sqrt{x}}\,dx.$$

ここで

$$\int\frac{(1-x)^2}{\sqrt{x}}\,dx=\int\frac{x^2-2x+1}{\sqrt{x}}\,dx=\int\left(x^{\frac{3}{2}}-2x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$$
$$=\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}.$$

よって

$$B\left(\frac{1}{2},3\right)=\int_0^1\frac{(1-x)^2}{\sqrt{x}}\,dx=\lim_{\alpha\to+0}\int_\alpha^1\frac{(1-x)^2}{\sqrt{x}}\,dx$$
$$=\lim_{\alpha\to+0}\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right]_\alpha^1$$
$$=\frac{2}{5}-\frac{4}{3}+2-\lim_{\alpha\to+0}\left(\frac{2}{5}\alpha^{\frac{5}{2}}-\frac{4}{3}\alpha^{\frac{3}{2}}+2\alpha^{\frac{1}{2}}\right)=\frac{16}{15}.$$

問 5. ガンマ関数 $\Gamma(s)$ は $s>0$ に対して収束するので，$\Gamma\left(\dfrac{1}{2}\right)=\displaystyle\int_0^\infty\frac{e^{-x}}{\sqrt{x}}\,dx$ は収束する．そこでまず,

$$\int_0^\infty\frac{e^{-x}}{\sqrt{x}}\,dx=2\int_0^\infty e^{-x^2}dx \qquad ①$$

を示す．広義積分 $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ は収束するので，

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \lim_{\alpha\to+0}\lim_{\beta\to\infty}\int_\alpha^\beta \frac{e^{-x}}{\sqrt{x}}\,dx \tag*{②}$$

である．$x = t^2\ (t > 0)$ と置換すると，$t = \sqrt{x}$，$dx = 2tdt$．また，x が α から β まで動くとき，t は $\sqrt{\alpha}$ から $\sqrt{\beta}$ まで動く．よって，置換積分法より，

$$\int_\alpha^\beta \frac{e^{-x}}{\sqrt{x}}\,dx = \int_{\sqrt{\alpha}}^{\sqrt{\beta}} \frac{e^{-t^2}}{t}\,2tdt = 2\int_{\sqrt{\alpha}}^{\sqrt{\beta}} e^{-x^2}dx \tag*{③}$$

となる．ゆえに，②と③より，

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = 2\lim_{\alpha\to+0}\lim_{\beta\to\infty}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$$

となり，①が示された．次に，

$$\int_0^\infty e^{-x^2}dx = \int_{-\infty}^0 e^{-x^2}dx \tag*{④}$$

を示す．①より，上式の左辺の広義積分は収束するので，

$$\int_0^\infty e^{-x^2}dx = \lim_{\beta\to 0}\int_0^\beta e^{-x^2}dx \tag*{⑤}$$

である．$t = -x$ とおくと，$dt = -dx$．また，x が 0 から β まで動くとき，t は 0 から $-\beta$ まで動く．よって，置換積分法より，

$$\int_0^\beta e^{-x^2}dx = \int_0^{-\beta} e^{-t^2}(-dt) = \int_{-\beta}^0 e^{-x^2}dx \tag*{⑥}$$

となる．ゆえに，⑤と⑥より，

$$\int_0^\infty e^{-x^2}dx = \lim_{\beta\to\infty}\int_{-\beta}^0 e^{-x^2}dx = \int_{-\infty}^0 e^{-x^2}dx$$

となり，④を得る．よって，広義積分の定義と④より，

$$\int_{-\infty}^\infty e^{-x^2}dx = \int_{-\infty}^0 e^{-x^2}dx + \int_0^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$$

が成り立つ．

問 6. (1) $0 < \alpha < \beta < \infty$ なる α，β に対して，部分積分法より，

$$\begin{aligned}\int_\alpha^\beta e^{-x}x^{s-1}\,dx &= \left[e^{-x}\cdot\frac{x^s}{s}\right]_\alpha^\beta - \int_\alpha^\beta \left(-e^{-x}\right)\cdot\frac{x^s}{s}\,dx \\ &= \frac{1}{s}\left(\frac{\beta^s}{e^\beta} - \frac{\alpha^s}{e^\alpha}\right) + \frac{1}{s}\int_\alpha^\beta e^{-x}x^s\,dx\end{aligned}$$

となる．よって，

$$\begin{aligned}\Gamma(s) &= \lim_{\alpha\to+0}\lim_{\beta\to\infty}\int_\alpha^\beta e^{-x}x^{s-1}\,dx \\ &= \frac{1}{s}\left(\lim_{\beta\to\infty}\frac{\beta^s}{e^\beta} - \lim_{\alpha\to+0}\frac{\alpha^s}{e^\alpha}\right) + \frac{1}{s}\lim_{\alpha\to+0}\lim_{\beta\to\infty}\int_\alpha^\beta e^{-x}x^s\,dx\end{aligned}$$

$$= \frac{1}{s} \lim_{\beta\to\infty} \frac{\beta^s}{e^\beta} + \frac{1}{s}\Gamma(s+1) \qquad ①$$

となる．$s \leqq n$ を満たす自然数 n を選ぶ．ロピタルの定理を繰り返し用いると，

$$\lim_{\beta\to\infty} \frac{\beta^n}{e^\beta} = \lim_{\beta\to\infty} \frac{n\beta^{n-1}}{e^\beta} = \lim_{\beta\to\infty} \frac{n(n-1)\beta^{n-2}}{e^\beta} = \cdots = \lim_{\beta\to\infty} \frac{n!}{e^\beta} = 0$$

なので，不等式

$$0 < \frac{\beta^s}{e^\beta} \leqq \frac{\beta^n}{e^\beta} \quad (\beta > 0)$$

より，$\lim\limits_{\beta\to\infty} \beta^s/e^\beta = 0$ となる．よって，①より，$\Gamma(s+1) = s\Gamma(s)$ を得る．また，示した漸化式を自然数 n に対して適用すると，

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-2) = (n-1)(n-2)\Gamma(n-3)\\ &= \cdots = (n-1)(n-2)\cdot\cdots\cdot 2\cdot 1\cdot\Gamma(1)\end{aligned}$$

となる．問 4 の (1) より，$\Gamma(1) = 1$ なので，$\Gamma(n) = (n-1)!$ を得る．

(2)　$0 < \alpha < \beta < 1$ を満たす α，β に対して，部分積分法より，

$$\begin{aligned}\int_\alpha^\beta x^p(1-x)^{q-2}\,dx &= \left[x^p\left\{-\frac{1}{q-1}(1-x)^{q-1}\right\}\right]_\alpha^\beta - \int_\alpha^\beta px^{p-1}\left\{-\frac{1}{q-1}(1-x)^{q-1}\right\}dx\\ &= -\frac{1}{q-1}\left\{\beta^p(1-\beta)^{q-1} - \alpha^p(1-\alpha)^{q-1}\right\}\\ &\qquad + \frac{p}{q-1}\int_\alpha^\beta x^{p-1}(1-x)^{q-1}dx.\end{aligned}$$

よって，

$$\begin{aligned}B(p+1,q-1) &= \lim_{\alpha\to+0}\lim_{\beta\to1-0}\int_\alpha^\beta x^p(1-x)^{q-2}dx\\ &= -\frac{1}{q-1}\left\{\lim_{\beta\to1-0}\beta^p(1-\beta)^{q-1} - \lim_{\alpha\to+0}\alpha^p(1-\alpha)^{q-1}\right\}\\ &\qquad + \frac{p}{q-1}\lim_{\alpha\to+0}\lim_{\beta\to1-0}\int_\alpha^\beta x^{p-1}(1-x)^{q-1}dx\\ &= \frac{p}{q-1}B(p,q)\end{aligned}$$

となる．ゆえに，$B(p,q) = \dfrac{q-1}{p}B(p+1,q-1)$ を得る．

(3)　(2) を繰り返し用いると

$$\begin{aligned}B(m,n) &= \frac{n-1}{m}B(m+1,n-1) = \frac{(n-1)(n-2)}{m(m+1)}B(m+2,n-2)\\ &= \frac{(n-1)(n-2)\cdots1}{m(m+1)\cdots(m+n-2)}B(m+n-1,1).\end{aligned}$$

ここで，

$$B(m+n-1,1) = \int_0^1 x^{m+n-2}\,dx = \left[\frac{1}{m+n-1}x^{m+n-1}\right]_0^1 = \frac{1}{m+n-1}.$$

なので，(1) を用いれば，

$$B(m,n) = \frac{(n-1)!}{m(m+1)\cdots(m+n-1)} = \frac{(m-1)!(n-1)!}{(m-1)!m(m+1)\cdots(m+n-1)}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}.$$

(4)　α, β は $0 < \alpha < \beta < 1$ を満たすとする．$x = \sin^2\theta$ $\left(0 < \theta < \frac{\pi}{2}\right)$ とおくと，$dx = 2\sin\theta\cos\theta d\theta$ で，x が α から β まで動くとき，θ は $\sin^{-1}\sqrt{\alpha}$ から $\sin^{-1}\sqrt{\beta}$ まで動く．よって

$$\int_\alpha^\beta x^{p-1}(1-x)^{q-1}dx = \int_{\sin^{-1}\sqrt{\alpha}}^{\sin^{-1}\sqrt{\beta}} \sin^{2p-2}\theta\left(1-\sin^2\theta\right)^{q-1} 2\sin\theta\cos\theta\,d\theta$$
$$= 2\int_{\sin^{-1}\sqrt{\alpha}}^{\sin^{-1}\sqrt{\beta}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$$

となる．ここで，$\alpha \to +0$, $\beta \to 1-0$ のとき，$\sin^{-1}\sqrt{\alpha} \to +0$, $\sin^{-1}\sqrt{\beta} \to \frac{\pi}{2} - 0$ なので，

$$B(p,q) = \lim_{\alpha\to+0}\lim_{\beta\to1-0}\int_\alpha^\beta x^{p-1}(1-x)^{q-1}dx$$
$$= 2\lim_{\alpha\to+0}\lim_{\beta\to1-0}\int_{\sin^{-1}\sqrt{\alpha}}^{\sin^{-1}\sqrt{\beta}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$$

を得る.

3.4　定積分の応用

問 1.　以下では，図形の面積を S とする．

(1)　曲線と直線の交点の x 座標は $x = 0, \pm 2$．曲線を表す関数は偶関数であることに注意すれば，

$$S = 2\int_0^2 \left\{3 - (x^2-1)(x^2-3)\right\} dx = 2\int_0^2 (4x^2 - x^4)\, dx = 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \frac{128}{15}.$$

(2)　曲線と直線の交点の x 座標は $x = \frac{1}{2}(3 \pm \sqrt{5})$．よって

$$\begin{aligned} S &= 2\int_0^{\frac{1}{2}(3-\sqrt{5})} \sqrt{x}\, dx + \int_{\frac{1}{2}(3-\sqrt{5})}^{\frac{1}{2}(3+\sqrt{5})} \left(\sqrt{x} - x + 1\right) dx \\ &= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^{\frac{1}{2}(3-\sqrt{5})} + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + x\right]_{\frac{1}{2}(3-\sqrt{5})}^{\frac{1}{2}(3+\sqrt{5})} \\ &= \frac{2}{3}\left(\frac{\sqrt{5}-1}{2}\right)^3 + \frac{2}{3}\left(\frac{\sqrt{5}+1}{2}\right)^3 - \frac{\sqrt{5}}{2} = \frac{5\sqrt{5}}{6}. \end{aligned}$$

(3)　曲線と直線の交点の x 座標は $x = 2$．よって

$$S = \int_0^2 \left(\frac{8}{x^2+4} - \frac{x}{2}\right) dx = \left[4\tan^{-1}\frac{x}{2} - \frac{x^2}{4}\right]_0^2 = 4\tan^{-1} 1 - 1 = \pi - 1.$$

(4)　t を消去すると，$y^2 = \frac{9}{8}x^3$．よって，$y = \pm\frac{3\sqrt{2}}{4}x^{\frac{3}{2}}$ となり，面積を求める図形は x 軸に関して対称．よって，

$$S = 2\int_0^1 \frac{3}{2\sqrt{2}}x^{\frac{3}{2}}\, dx = \frac{3}{\sqrt{2}}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = \frac{3\sqrt{2}}{5}.$$

別解．　$x = 1$ のとき $t = \pm\frac{1}{\sqrt{2}}$．面積を求める図形が x 軸に関して対称であることに注意すると，

$$S = 2\int_0^1 y\, dx = 2\int_0^{\frac{1}{\sqrt{2}}} y \cdot \frac{dx}{dt}\, dt = 24\int_0^{\frac{1}{\sqrt{2}}} t^4\, dt = 24\left[\frac{t^5}{5}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{3\sqrt{2}}{5}.$$

問 2.　図形の面積を S とすると，

$$\begin{aligned} S &= \int_0^{2\pi a} y\, dx = \int_0^{2\pi} y \cdot \frac{dx}{dt}\, dt = \int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\, dt \\ &= a^2\int_0^{2\pi} (1-\cos t)^2\, dt = a^2\int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt \\ &= \frac{a^2}{2}\int_0^{2\pi} (3 - 4\cos t + \cos 2t)\, dt = \frac{a^2}{2}\left[3t - 4\sin t + \frac{1}{2}\sin 2t\right]_0^{2\pi} = 3\pi a^2. \end{aligned}$$

問 3.　以下では，図形の面積を S とする．

(1)　教科書の図 3.3 より，図形は x 軸と y 軸に関して対称．よって，第一象限の部分の面積を 4 倍して，

$$S = \frac{4}{2}\int_0^{\frac{\pi}{4}} 2a^2\cos 2\theta\, d\theta = 4a^2\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = 2a^2.$$

(2)　教科書の図 3.3 より，図形は x 軸と y 軸に関して対称．よって，第一象限の部分の面積を 4 倍して，

$$S = \frac{4}{2}\int_0^{\frac{\pi}{2}} a^2\sin^2 2\theta\, d\theta = 2a^2\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 4\theta)\, d\theta = a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}a^2.$$

(3)　教科書の図 3.3 より，

$$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\theta^2 d\theta = \frac{a^2}{2}\left[\frac{\theta^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{48}a^2.$$

問 4.　以下では，曲線の長さを L とする．

(1)　教科書の図 3.3 より，

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\{a(1-\cos t)^2 + (a\sin t)^2}\,dt = \sqrt{2}\,a\int_0^{2\pi}\sqrt{1-\cos t}\,dt \\ &= \sqrt{2}\,a\int_0^{2\pi}\sqrt{2\sin^2\frac{t}{2}}\,dt = 2a\int_0^{2\pi}\sin\frac{t}{2}\,dt = 2a\left[-2\cos\frac{t}{2}\right]_0^{2\pi} \\ &= -4a\big(\cos\pi - \cos 0\big) = 8a. \end{aligned}$$

(2)　$f(x) = \sqrt{a^2 - x^2}$ とおくと，$f'(x) = -\dfrac{x}{\sqrt{a^2-x^2}}$，$1 + f'(x)^2 = \dfrac{a^2}{a^2-x^2}$．よって

$$\begin{aligned} L &= \int_{-a}^{a}\sqrt{\frac{a^2}{a^2-x^2}}\,dx = a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}} = a\left[\sin^{-1}\frac{x}{a}\right]_{-a}^{a} \\ &= a\Big\{\sin^{-1}1 - \sin^{-1}(-1)\Big\} = a\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \pi a. \end{aligned}$$

(3)　$f(\theta) = a(1+\cos\theta)$ とおくと，$f'(\theta) = -a\sin\theta$．よって

$$f(\theta)^2 + f'(\theta)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta = 2a^2(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2}.$$

教科書の図 3.3 より，曲線は x 軸に関して対称．よって

$$L = 2\int_0^{\pi}\sqrt{4a^2\cos^2\frac{\theta}{2}}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

問 5.　以下では，立体の体積を V とする．

(1)　立体を x 軸に垂直な平面で切った断面積を $S(x)$ とすると，

$$S(x) = \frac{1}{2}\cdot b\cdot\left(1-\frac{x}{a}\right)\cdot c\left(1-\frac{x}{a}\right) = \frac{bc}{2}\left(1-\frac{x}{a}\right)^2.$$

よって

$$V = \int_0^a S(x)\,dx = \frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[-\frac{a}{3}\left(1-\frac{x}{a}\right)^3\right]_0^a = \frac{abc}{6}.$$

(2)　正三角錐の底面の面積は $\dfrac{1}{2}\cdot a\cdot\dfrac{\sqrt{3}}{2}a = \dfrac{\sqrt{3}}{4}a^2$．底面から頂点までの高さ h は，ピタゴラスの定理より，

$$h^2 + \left(\frac{1}{3}\cdot\frac{\sqrt{3}}{2}a\right)^2 = \left(\frac{\sqrt{3}}{2}a\right)^2 \qquad \therefore\quad h = \frac{\sqrt{6}}{3}a.$$

底面からの高さが x である底面と平行な平面でこの正三角錐を切った部分の断面は，1 辺の長さが $\dfrac{h-x}{x}a$ の正三角形になるので，その面積 $S(x)$ は

$$S(x) = \frac{\sqrt{3}}{4}\left(\frac{h-x}{h}a\right)^2.$$

よって

$$V=\int_0^h S(x)\,dx=\frac{\sqrt{3}}{4}\,a^2\int_0^h\left(1-\frac{x}{h}\right)^2dx=\frac{\sqrt{3}'}{4}\,a^2\left[-\frac{h}{3}\left(1-\frac{x}{h}\right)^3\right]_0^h=\frac{\sqrt{2}}{12}\,a^3.$$

問 6.　楕円球の対称性より，その体積を V とすると

$$V=2\pi\int_0^a\frac{b^2}{a^2}\left(a^2-x^2\right)dx=\frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{4\pi}{3}\,ab^2.$$

問 7.　以下では，回転体の体積を V とする．

(1)　教科書の図 3.3 より，図形の対称性を考慮すると，

$$V=2\pi\int_0^a\left\{\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\right\}^2dx=\frac{\pi a^2}{2}\int_0^a\left(e^{\frac{2}{a}x}+e^{-\frac{2}{a}x}+2\right)dx$$
$$=\frac{\pi a^2}{2}\left[\frac{a}{2}e^{\frac{2}{a}x}-\frac{a}{2}e^{-\frac{2}{a}x}+2x\right]_0^a=\frac{\pi a^3}{4}\left(e^2-e^{-2}+4\right).$$

(2)　$f(x)=b+\sqrt{a^2-x^2}$，$g(x)=b-\sqrt{a^2-x^2}$ とおくと，図形の対称性より，

$$\begin{aligned}V&=2\pi\int_0^a f(x)^2\,dx-2\pi\int_0^a g(x)^2\,dx\\&=2\pi\int_0^a\{f(x)+g(x)\}\cdot\{f(x)-g(x)\}\,dx\\&=2\pi\int_0^a 2b\cdot2\sqrt{a^2-x^2}dx=8\pi b\int_0^a\sqrt{a^2-x^2}\,dx\\&=8\pi b\left[\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right)\right]_0^a\quad(\text{例 3.1.3 の公式})\\&=8\pi b\cdot\frac{\pi a^2}{4}=2\pi^2a^2b.\end{aligned}$$

(3)　教科書 80 頁の図 3.3 より，

$$V=2\pi\int_0^{\pi a}y^2\,dx.$$

ここで，$x=a(t-\sin t)$，$y=a(1-\cos t)$ なので，$dx=a(1-\cos t)\,dt$，$y^2=a^2(1-\cos t)^2$．また，x が 0 から πa まで動くとき，t は 0 から π まで動く．よって

$$V=2\pi\int_0^\pi a^2(1-\cos t)^2\cdot a(1-\cos t)\,dt=2\pi a^3\int_0^\pi(1-\cos t)^3\,dt.$$

ここで

$$\begin{aligned}(1-\cos t)^3&=1-3\cos t+3\cos^2 t-\cos^3 t\\&=1-3\cos t+3\cdot\frac{1+\cos 2t}{2}-\frac{\cos 3t+3\cos t}{4}\\&=\frac{5}{2}-\frac{15}{4}\cos t+\frac{3}{2}\cos 2t-\frac{1}{4}\cos 3t.\end{aligned}$$

よって

$$V=2\pi a^3\left[\frac{5}{2}t-\frac{15}{4}\sin t+\frac{3}{4}\sin 2t-\frac{1}{12}\sin 3t\right]_0^\pi=2\pi a^3\cdot\frac{5\pi}{2}=5\pi^2a^3.$$

演習問題 3

1. (1) $\displaystyle \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int \left(x - 2 + \frac{1}{x}\right) dx = \frac{x^2}{2} - 2x + \log|x|$

(2) $\displaystyle \int (1-x)\sqrt[3]{x}\,dx = \int \left(x^{\frac{1}{3}} - x^{\frac{4}{3}}\right) dx = \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}} = \frac{3}{28}x(7-4x)\sqrt[3]{x}$

(3) $\displaystyle \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$ より，

$$\begin{aligned}\int \left(\frac{2}{1-x^2} + \frac{3}{\sqrt{1-x^2}}\right) dx &= \int \left(\frac{1}{1-x} + \frac{1}{1+x} + \frac{3}{\sqrt{1-x^2}}\right) dx \\ &= -\log|1-x| + \log|1+x| + 3\sin^{-1} x \\ &= \log\left|\frac{1+x}{1-x}\right| + 3\sin^{-1} x\end{aligned}$$

(4) $\sqrt[3]{x+1} = t$ とおくと，$x = t^3 - 1$，$dx = 3t^2 dt$．よって

$$\begin{aligned}\int x\sqrt[3]{x+1}\,dx &= \int (t^3-1)t \cdot 3t^2 dt = 3\int (t^6 - t^3)\,dt = 3\left(\frac{t^7}{7} - \frac{t^4}{4}\right) \\ &= \frac{3}{28}t^3(4t^3-7)t = \frac{3}{28}(x+1)(4x-3)\sqrt[3]{x+1}.\end{aligned}$$

別解. $\displaystyle \int x\sqrt[3]{x+1}\,dx = \int \{(x+1)-1\}\sqrt[3]{x+1}\,dx = \int \left\{(x+1)^{\frac{4}{3}} - (x+1)^{\frac{1}{3}}\right\} dx$

$$= \frac{3}{7}(x+1)^{\frac{7}{3}} - \frac{3}{4}(x+1)^{\frac{4}{3}} = \frac{3}{28}(x+1)(4x-3)\sqrt[3]{x+1}.$$

(5) $\displaystyle \frac{2x+3}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ とおくと，

$$\begin{aligned}2x+3 &= A(x+1)^2 + B(x+2)(x+1) + C(x+2) \\ &= (A+B)x^2 + (2A+3B+C)x + (A+2B+2C).\end{aligned}$$

両辺の係数を比較すると，$A+B=0$，$2A+3B+C=2$，$A+2B+2C=3$．これを解いて，$A=-1$，$B=1$，$C=1$．よって

$$\frac{2x+3}{(x+2)(x+1)^2} = -\frac{1}{x+2} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

と部分分数分解できる．よって

$$\begin{aligned}\int \frac{2x+3}{(x+2)(x+1)^2}\,dx &= \int \left\{-\frac{1}{x+2} + \frac{1}{x+1} + \frac{1}{(x+1)^2}\right\} dx \\ &= -\log|x+2| + \log|x+1| - \frac{1}{x+1} \\ &= \log\left|\frac{x+1}{x+2}\right| - \frac{1}{x+1}.\end{aligned}$$

(6) $\displaystyle \frac{x^5 - x^2 + 1}{x^3 - 1} = x^2 + \frac{1}{x^3-1} = x^2 + \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right)$ なので，

$$\int \frac{x^5 - x^2 + 1}{x^3-1}\,dx$$

$$= \int \left\{ x^2 + \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} \right) \right\} dx$$

$$= \int \left\{ x^2 + \frac{1}{3} \cdot \frac{1}{x-1} - \frac{1}{6} \cdot \frac{2x+1}{x^2+x+1} - \frac{1}{2} \cdot \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} dx$$

$$= \frac{x^3}{3} + \frac{1}{3}\log|x-1| - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)$$

$$= \frac{x^3}{3} + \frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

(7)　$x^2 = X$ とおいて，与式を X について部分分数分解すると，

$$\frac{x^2+1}{x^4+2x^2-3} = \frac{x^2+1}{(x^2+3)(x^2-1)}$$

$$= \frac{X+1}{(X+3)(X-1)} = \frac{1}{2}\left(\frac{1}{X-1} + \frac{1}{X+3} \right)$$

$$= \frac{1}{2}\left(\frac{1}{x^2-1} + \frac{1}{x^2+3} \right)$$

となる．ゆえに

$$\int \frac{x^2+1}{x^4+2x^2-3}\,dx = \frac{1}{2}\int \left(\frac{1}{x^2-1} + \frac{1}{x^2+3} \right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} \right)$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}.$$

(8)　$x = e^t$ とくと，$t = \log x$，$dx = e^t dt$．よって

$$\int x^2 \log x\,dx = \int \left(e^t\right)^2 t \cdot e^t dt = \int te^{3t}dt = t\left(\frac{1}{3}e^{3t}\right) - \int \frac{1}{3}e^{3t}dt$$

$$= \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} = \frac{x^3}{9}(3\log x - 1).$$

別解．　$\int x^2 \log x\,dx = \frac{x^3}{3}\log x - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2\,dx$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} = \frac{x^3}{9}(3\log x - 1).$$

(9)　$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$ より，

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x).$$

(10)　$x^2 = t$ とおくと，$2xdx = dt$．よって

$$\int x^3 e^{-x^2}dx = \frac{1}{2}\int te^{-t}dt = \frac{1}{2}\left\{ t\left(-e^{-t}\right) - \int \left(-e^{-t}\right)dt \right\}$$

$$= \frac{1}{2}\left(-te^{-t} - e^{-t}\right) = -\frac{\left(x^2+1\right)e^{-x^2}}{2}$$

(11) $\log x = t$ とおくと，$x = e^t$，$dx = e^t dt$．よって

$$\int \frac{dx}{x(\log x)^3} = \int \frac{e^t}{e^t \cdot t^3}\,dt = \int t^{-3} dt = -\frac{1}{2}t^{-2} = -\frac{1}{2(\log x)^2}.$$

別解. $\displaystyle \int \frac{dx}{x(\log x)^3} = \int (\log x)'(\log x)^{-3}\,dx = -\frac{1}{2}(\log x)^{-2} = -\frac{1}{2(\log x)^2}.$

(12) $\sqrt{e^{2x}+1} = t$ とおくと，$e^{2x}+1 = t^2$，$e^{2x}dx = tdt$．よって，$dx = \dfrac{t}{t^2-1}\,dt$．ゆえに

$$\begin{aligned}
\int \sqrt{e^{2x}+1}\,dx &= \int \frac{t^2}{t^2-1}\,dt = \int \left(1 + \frac{1}{t^2-1}\right) dt \\
&= \int \left\{1 + \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)\right\} dt \\
&= t + \frac{1}{2}(\log|t-1| - \log|t+1|) \\
&= t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| \\
&= \sqrt{e^{2x}+1} + \frac{1}{2}\log\frac{\sqrt{e^{2x}+1}-1}{\sqrt{e^{2x}+1}+1} \\
&= \sqrt{e^{2x}+1} + \frac{1}{2}\log\frac{(\sqrt{e^{2x}+1}-1)^2}{e^{2x}} \\
&= \sqrt{e^{2x}+1} + \log(\sqrt{e^{2x}+1}-1) - x.
\end{aligned}$$

(13) $\sin^{-1} x = t$ とおくと，$x = \sin t$ $\left(-\dfrac{\pi}{2} \leqq t \leqq \dfrac{\pi}{2}\right)$，$dx = \cos t\,dt$．よって

$$\begin{aligned}
\int (\sin^{-1} x)^2\,dx &= \int t^2 \cos t\,dt = t^2 \sin t - 2\int t \sin t\,dt \\
&= t^2 \sin t + 2t\cos t - 2\int \cos t\,dt \\
&= t^2 \sin t + 2t\sqrt{1-\sin^2 t} - 2\sin t \\
&= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x.
\end{aligned}$$

(14) $\sqrt{x-1} = t$ とおくと，$x = t^2+1$，$dx = 2t\,dt$．よって

$$\begin{aligned}
\int \frac{dx}{x+\sqrt{x-1}} &= \int \frac{2t}{t^2+t+1}\,dt = \int \frac{(2t+1)-1}{t^2+t+1}\,dt \\
&= \int \frac{2t+1}{t^2+t+1}\,dt - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right) \\
&= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}} \\
&= \log\left(x+\sqrt{x-1}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x-1}+1}{\sqrt{3}}.
\end{aligned}$$

(15)　$\log x = t$ とおくと，$\dfrac{dx}{x} = dt$．よって

$$\int \frac{\sin(\log x)}{x}\,dx = \int \sin t\,dt = -\cos t = -\cos(\log x).$$

別解．$\displaystyle\int \frac{\sin(\log x)}{x}\,dx = \int (\log x)' \sin(\log x)\,dx = -\cos(\log x).$

(16)　$\sqrt{\dfrac{x}{x-1}} = t$ とおくと，$x = \dfrac{t^2}{t^2-1}$，$dx = -\dfrac{2t}{(t^2-1)^2}\,dt$．よって

$$\begin{aligned}\int \frac{1}{x}\sqrt{\frac{x}{x-1}}\,dx &= -\int \frac{t^2-1}{t^2}\cdot\frac{2t^2}{(t^2-1)^2}\,dt = -2\int \frac{dt}{t^2-1}\\ &= -\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt = \log|t+1| - \log|t-1|\\ &= \log\left|\frac{t+1}{t-1}\right| = \log\left|\frac{\sqrt{\frac{x}{x-1}}+1}{\sqrt{\frac{x}{x-1}}-1}\right|\\ &= \log\left|\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}\right| = 2\log\left(\sqrt{x}+\sqrt{x-1}\right).\end{aligned}$$

(17)　$x = \sqrt{2}\tan t\ \left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$ とおくと，

$$dx = \frac{\sqrt{2}}{\cos^2 t}\,dt, \quad \left(x^2+2\right)^{\frac{5}{2}} = \left(2\tan^2 t + 2\right)^{\frac{5}{2}} = 4\sqrt{2}\left(1+\tan^2 t\right)^{\frac{5}{2}} = \frac{4\sqrt{2}}{\cos^5 t}.$$

よって

$$\begin{aligned}\int \frac{x^2}{(x^2+2)^{\frac{5}{2}}}\,dx &= \int 2\tan^2 t\cdot\frac{\cos^5 t}{4\sqrt{2}}\cdot\frac{\sqrt{2}}{\cos^2 t}\,dt = \frac{1}{2}\int \tan^2 t\cos^3 t\,dt\\ &= \frac{1}{2}\int \sin^2 t\cos t\,dt = \frac{1}{6}\sin^3 t\\ &= \frac{1}{6}\left(\frac{\tan^2 t}{\tan^2 t + 1}\right)^{\frac{3}{2}} = \frac{1}{6}\left(\frac{x^2}{x^2+2}\right)^{\frac{3}{2}}.\end{aligned}$$

(18)　$x = \sin t\ \left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$ とおくと，$dx = \cos t\,dt$．よって

$$\int \frac{dx}{(1-x^2)^{\frac{5}{2}}} = \int \frac{\cos t}{(1-\sin^2 t)^{\frac{5}{2}}}\,dt = \int \frac{\cos t}{\cos^5 t}\,dt = \int \frac{dt}{\cos^4 t}.$$

そこで，$\tan t = y$ とおくと，$\dfrac{dt}{\cos^2 t} = dy$，$\dfrac{1}{\cos^4 t} = \dfrac{1+\tan^2 t}{\cos^2 t}$．よって

$$\int \frac{dt}{\cos^4 t} = \int \left(1+y^2\right)dy = y + \frac{y^3}{3} = \tan t + \frac{\tan^3 t}{3}.$$

さて，$\cos^2 t = 1 - \sin^2 t = 1 - x^2$，$\cos t \geqq 0$ なので，$\cos t = \sqrt{1-x^2}$．よって，$\tan t = \dfrac{x}{\sqrt{1-x^2}}$．
ゆえに

$$\int \frac{dx}{(1-x^2)^{\frac{5}{2}}} = \tan t + \frac{\tan^3 t}{3} = \frac{x(3-2x^2)}{3\,(1-x^2)^{\frac{3}{2}}}.$$

(19) $\sqrt{1+x+x^2}=t-x$ とおくと，

$$x=\frac{t^2-1}{2t+1},\quad dx=\frac{2(t^2+t+1)}{(2t+1)^2}\,dt,\quad \sqrt{1+x+x^2}=\frac{t^2+t+1}{2t+1},\quad 1-x=\frac{-t^2+2t+2}{2t+1}.$$

よって

$$\begin{aligned}
\int\frac{dx}{(1-x)\sqrt{1+x+x^2}} &= \int\frac{2t+1}{-t^2+2t+2}\cdot\frac{2t+1}{t^2+t+1}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt\\
&= -2\int\frac{dt}{t^2-2t-2}\,dt=-2\int\frac{dt}{(t-1)^2-3}\,dt\\
&= -2\cdot\frac{1}{2\sqrt{3}}\log\left|\frac{t-1-\sqrt{3}}{t-1+\sqrt{3}}\right| \quad (\text{例 }8)\\
&= \frac{1}{\sqrt{3}}\log\left|\frac{t-1+\sqrt{3}}{t-1-\sqrt{3}}\right|=\frac{1}{\sqrt{3}}\log\left|\frac{x+\sqrt{1+x+x^2}-1+\sqrt{3}}{x+\sqrt{1+x+x^2}-1-\sqrt{3}}\right|.
\end{aligned}$$

(20) $\sqrt{2+x-x^2}=\sqrt{(2-x)(x+1)}=(2-x)\sqrt{\dfrac{x+1}{2-x}}$ より，$\sqrt{\dfrac{x+1}{2-x}}=t$ とおくと，

$$x=\frac{2t^2+1}{t^2+1},\quad \sqrt{2+x-x^2}=\frac{3t}{t^2+1},\quad dx=\frac{6t}{(t^2+1)^2}dt.$$

よって

$$\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{t^2+1}{3t}\cdot\frac{6t}{(t^2+1)^2}\,dt=2\int\frac{dt}{t^2+1}=2\tan^{-1}t=2\tan^{-1}\sqrt{\frac{x+1}{2-x}}.$$

別解. $\sqrt{2+x-x^2}=\sqrt{(2-x)(x+1)}=(x+1)\sqrt{\dfrac{2-x}{x+1}}$ より，$\sqrt{\dfrac{2-x}{x+1}}=t$ とおくと，

$$x=-\frac{t^2-1}{t^2+1},\quad \sqrt{2+x-x^2}=\frac{2t}{t^2+1},\quad dx=-\frac{4t}{(t^2+1)^2}\,dt.$$

よって

$$\begin{aligned}
\int\frac{dx}{\sqrt{2+x-x^2}} &= \int\frac{t^2+1}{2t}\cdot\frac{-4t}{(t^2+1)^2}\,dt=-2\int\frac{dt}{t^2+1}\\
&= -2\tan^{-1}t=-2\tan^{-1}\sqrt{\frac{2-x}{x+1}}.
\end{aligned}$$

別解. $\displaystyle\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}=\sin^{-1}\frac{2}{3}\left(x-\frac{1}{2}\right)=\sin^{-1}\frac{2x-1}{3}.$

(21) $\tan\dfrac{x}{2}=t$ とおくと，

$$dx=\frac{2}{1+t^2}\,dt,\quad \cos x=\frac{1-t^2}{1+t^2},\quad 1+2\cos x=\frac{3-t^2}{1+t^2}.$$

よって

$$\int\frac{dx}{1+2\cos x}=\int\frac{1+t^2}{3-t^2}\cdot\frac{2}{1+t^2}\,dt=-2\int\frac{dt}{t^2-3}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{3}} \int \left(\frac{1}{t+\sqrt{3}} - \frac{1}{t-\sqrt{3}} \right) dt \\
&= \frac{1}{\sqrt{3}} \left(\log|t+\sqrt{3}| - \log|t-\sqrt{3}| \right) \\
&= \frac{1}{\sqrt{3}} \log\left| \frac{t+\sqrt{3}}{t-\sqrt{3}} \right| = \frac{1}{\sqrt{3}} \log\left| \frac{\tan\frac{x}{2}+\sqrt{3}}{\tan\frac{x}{2}-\sqrt{3}} \right|.
\end{aligned}
$$

2. $n \neq 0, 1$ とする．部分積分法を用いて計算すると，

$$
\begin{aligned}
I_n &= \int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx = -\int \sin^{n-1} x(\cos x)'\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n
\end{aligned}
$$

となる．よって，漸化式

$$
I_n = \frac{1}{n}\left\{-(\sin x)^{n-1}\cos x + (n-1)I_{n-2}\right\} \quad (n \neq 0, 1) \tag{$*$}
$$

を得る．

$n = 1$ のときは,

$$
I_1 = \int \sin x\,dx = -\cos x.
$$

一方，(*) で $n = 1$ とおくと，

$$
I_1 = \frac{1}{1}\left\{-(\sin x)^0\cos x + (1-1)I_{-1}\right\} = -\cos x.
$$

よって，(*) は $n = 1$ のときも成り立つ．以上より，漸化式 (*) は 0 以外のすべての整数 n で成り立つ．

次に，(*) に $n = 2, 4$ を代入すると，

$$
I_2 = -\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0, \quad I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2.
$$

さらに，I_0 の定義より,

$$
I_0 = \int (\sin x)^0\,dx = \int dx = x.
$$

以上より,

$$
\begin{aligned}
\int \sin^4 x\,dx = I_4 &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\cos x + \frac{x}{2}\right) \\
&= \frac{1}{8}\left\{-2\sin^3 x\cos x - 3\sin x\cos x + 3x\right\} \\
&= -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x.
\end{aligned}
$$

また，(*) に $n = -2$ を代入すると，

$$
I_{-2} = \frac{1}{2}\cdot\frac{\cos x}{\sin^3 x} + \frac{3}{2}I_{-4}.
$$

さらに，I_{-2} の定義より，

$$I_{-2} = \int \frac{1}{\sin^2 x} dx = -\frac{1}{\tan x}.$$

よって

$$\int \frac{dx}{\sin^4 x} = I_{-4} = \frac{2}{3}\left(-\frac{1}{2}\frac{\cos x}{\sin^3 x} + I_{-2}\right) = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3\tan x}$$
$$= -\frac{\cos x}{3\sin^3 x}(1 + 2\sin^2 x).$$

3. (1) $\displaystyle\int_1^{e^2} \frac{\log x + 1}{x}\, dx = \left[\frac{1}{2}(\log x + 1)^2\right]_1^{e^2} = \frac{1}{2}(9 - 1) = 4.$

(2) $\sqrt{x+1} = t$ とおくと，$x + 1 = t^2$，$dx = 2tdt$．また，x が 0 から 1 まで動くとき，t は 1 から $\sqrt{2}$ まで動く．よって

$$\int_0^1 x\sqrt{x+1}\, dx = \int_1^{\sqrt{2}} \left(t^2 - 1\right) \cdot t \cdot (2t\, dt) = 2\int_1^{\sqrt{2}} \left(t^4 - t^2\right) dt$$
$$= 2\left[\frac{t^5}{5} - \frac{t^3}{3}\right]_1^{\sqrt{2}} = \frac{2}{5}\left(4\sqrt{2} - 1\right) - \frac{2}{3}\left(2\sqrt{2} - 1\right)$$
$$= \frac{4}{15}\left(1 + \sqrt{2}\right).$$

別解. $\displaystyle\int x\sqrt{x+1}\, dx = \int \{(x+1) - 1\}\sqrt{x+1}\, dx = \int \left\{(x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}}\right\} dx$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}.$$

よって

$$\int_0^1 x\sqrt{x+1}\, dx = \left[\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^1 = \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}$$
$$= \frac{4}{15}\left(1 + \sqrt{2}\right).$$

(3) $\sqrt{x+1} = t$ とおくと，$x + 1 = t^2$，$dx = 2tdt$．また，x が 0 から 1 まで動くとき，t は 1 から $\sqrt{2}$ まで動く．よって

$$\int_0^1 \frac{x^2}{\sqrt{x+1}}\, dx = \int_1^{\sqrt{2}} \frac{\left(t^2 - 1\right)^2}{t} \cdot 2t\, dt = 2\int_1^{\sqrt{2}} \left(t^4 - 2t^2 + 1\right) dt$$
$$= 2\left[\frac{t^5}{5} - \frac{2}{3}t^3 + t\right]_1^{\sqrt{2}}$$
$$= \frac{2}{5}\left(4\sqrt{2} - 1\right) - \frac{4}{3}\left(2\sqrt{2} - 1\right) + 2\left(\sqrt{2} - 1\right)$$
$$= \frac{2}{15}\left(7\sqrt{2} - 8\right).$$

別解. $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\, dx = \int \frac{(x+1)^2 - 2(x+1) + 1}{\sqrt{x+1}}\, dx$

$$= \int \left\{(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + (x+1)^{-\frac{1}{2}}\right\} dx$$
$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}}.$$

これより

$$\int_0^1 \frac{x^2}{\sqrt{x+1}}\,dx = \left[\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}}\right]_0^1$$
$$= \frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} + 2\sqrt{2} - \frac{2}{5} + \frac{4}{3} - 2$$
$$= \frac{14\sqrt{2}}{15} - \frac{16}{15} = \frac{2}{15}\left(7\sqrt{2} - 8\right).$$

(4) $\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_0^2 (4-x^2)'(4-x^2)^{\frac{1}{2}}\,dx = -\frac{1}{2}\left[\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2 = \frac{8}{3}.$

(5) $\displaystyle\int_0^1 \frac{dx}{x^2+x+1} = \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{9}\pi.$$

(6) $\displaystyle\int_1^2 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}} = \frac{1}{2}\int_1^2 \left(\sqrt{x+1}+\sqrt{x-1}\right)dx$

$$= \frac{1}{2}\left[\frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 = \sqrt{3} - \frac{2}{3}\sqrt{2} + \frac{1}{3}.$$

(7) $x = \tan\theta$ とおくと，$dx = \dfrac{d\theta}{\cos^2\theta}$．また，$x$ が 0 から 1 まで動くとき，θ は 0 から $\dfrac{\pi}{4}$ まで動く．よって

$$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(\tan^2\theta+1)^2}\cdot\frac{d\theta}{\cos^2\theta} = \int_0^{\frac{\pi}{4}} \sin^2\theta\cos^2\theta\,d\theta.$$

ゆえに

$$\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}\cdot\frac{1-\cos 4\theta}{2} = \frac{1}{8}(1-\cos 4\theta)$$

より,

$$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} (1-\cos 4\theta)\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{32}.$$

別解. $\displaystyle\frac{x^2}{(x^2+1)^3} = \frac{(x^2+1)-1}{(x^2+1)^3} = \frac{1}{(x^2+1)^2} - \frac{1}{(x^2+1)^3}$ と部分分数分解できるので,

$$\int \frac{x^2}{(x^2+1)^3}\,dx = \int \frac{dx}{(x^2+1)^2} - \int \frac{dx}{(x^2+1)^3}.$$

ここで,

$$\int \frac{dx}{x^2+1} = \frac{x}{x^2+1} + \int \frac{2x^2}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + 2\int \frac{(x^2+1)-1}{(x^2+1)^2}\,dx$$
$$= \frac{x}{x^2+1} + 2\int \frac{dx}{x^2+1} - 2\int \frac{dx}{(x^2+1)^2}$$

より

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1}x\right).$$

また

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{(x^2+1)^2} + \int \frac{4x^2}{(x^2+1)^3}\,dx = \frac{x}{(x^2+1)^2} + 4\int \frac{(x^2+1)-1}{(x^2+1)^3}\,dx$$
$$= \frac{x}{(x^2+1)^2} + 4\int \frac{dx}{(x^2+1)^2} - 4\int \frac{dx}{(x^2+1)^3}$$

より

$$\int \frac{dx}{(x^2+1)^3} = \frac{1}{4}\left\{\frac{x}{(x^2+1)^2} + 3\int \frac{dx}{(x^2+1)^2}\right\}$$
$$= \frac{x}{4(x^2+1)^2} + \frac{3}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right).$$

ゆえに

$$\int \frac{x^2}{(x^2+1)^3}\,dx = \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2} - \frac{3}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right)$$
$$= \frac{1}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2}.$$

よって

$$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \left[\frac{1}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2}\right]_0^1$$
$$= \frac{1}{8}\tan^{-1} 1 = \frac{1}{8}\cdot\frac{\pi}{4} = \frac{\pi}{32}.$$

(8)　$\sqrt{x^2+x+1} = t - x$ とおくと，

$$x = \frac{t^2-1}{2t+1},\quad \sqrt{x^2+x+1} = \frac{t^2+t+1}{2t+1},\quad dx = \frac{2(t^2+t+1)}{(2t+1)^2}\,dt.$$

また，x が 1 から 2 まで動くとき，t は $1+\sqrt{3}$ から $2+\sqrt{7}$ まで動く．よって

$$\int_1^2 \frac{dx}{x\sqrt{x^2+x+1}} = \int_{1+\sqrt{3}}^{2+\sqrt{7}} \frac{1}{\dfrac{t^2-1}{2t+1}\cdot\dfrac{t^2+t+1}{2t+1}}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt$$
$$= \int_{1+\sqrt{3}}^{2+\sqrt{7}} \frac{2}{t^2-1}\,dt = \left[\log\left|\frac{t-1}{t+1}\right|\right]_{1+\sqrt{3}}^{2+\sqrt{7}}$$
$$= \log\frac{1+\sqrt{7}}{3+\sqrt{7}} - \log\frac{\sqrt{3}}{2+\sqrt{3}}.$$

(9)　$\log x = t$ とおくと，$\dfrac{dx}{x} = dt$，$x = e^t$．また，x が 1 から e まで動くとき，t は 0 から 1 まで動く．そこで，$I := \displaystyle\int_1^e \cos(\log x)\,dx$ とおくと，

$$I = \int_0^1 e^t\cos t\,dt = \Big[e^t\sin t\Big]_0^1 - \int_0^1 e^t\sin t\,dt$$
$$= e\sin 1 - \left\{\Big[e^t(-\cos t)\Big]_0^1 - \int_0^1 e^t(-\cos t)\,dt\right\}$$
$$= e(\sin 1 + \cos 1) - 1 - I.$$

ゆえに，$I=\dfrac{e}{2}(\sin 1+\cos 1)-\dfrac{1}{2}$.

別解.　部分積分すると

$$\begin{aligned}\int\cos(\log x)\,dx &= x\cos(\log x)+\int\sin(\log x)\,dx\\ &= x\cos(\log x)+x\sin(\log x)-\int\cos(\log x)\,dx.\end{aligned}$$

よって

$$\int\cos(\log x)\,dx=\frac{x}{2}\Big\{\cos(\log x)+\sin(\log x)\Big\}.$$

ゆえに

$$\int_1^e\cos(\log x)\,dx=\left[\,\frac{x}{2}\,\{\cos(\log x)+\sin(\log x)\}\right]_1^e=\frac{e}{2}(\cos 1+\sin 1)-\frac{1}{2}.$$

(10)　$\displaystyle\int x^2\tan^{-1}x\,dx=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x^3}{x^2+1}\,dx$

$$\begin{aligned}&=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{x^2+1}\right)dx\\ &=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\log(x^2+1).\end{aligned}$$

よって

$$\int_0^1 x^2\tan^{-1}x\,dx=\left[\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\log(x^2+1\,)\right]_0^1=\frac{\pi}{12}+\frac{1}{6}(\log 2-1)$$

(11)　$\tan\dfrac{x}{2}=t$ とおくと，$\sin x=\dfrac{2t}{1+t^2}$，$dx=\dfrac{2dt}{1+t^2}$．また，x が 0 から $\dfrac{\pi}{3}$ まで動くとき，t は 0 から $\dfrac{1}{\sqrt{3}}$ まで動く．よって

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\frac{dx}{1+\sin x}&=\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=2\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{(t+1)^2}\\ &=2\left[-\frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}}=\sqrt{3}-1\end{aligned}$$

別解.　$\displaystyle\int_0^{\frac{\pi}{3}}\frac{dx}{1+\sin x}=\int_0^{\frac{\pi}{3}}\frac{1-\sin x}{\cos^2 x}\,dx=\left[\tan x-\frac{1}{\cos x}\right]_0^{\frac{\pi}{3}}=\sqrt{3}-1.$

(12)　部分積分法より，

$$\begin{aligned}\int_0^{\frac{\pi}{2}}x^2\cos x\,dx&=\Big[x^2\sin x\Big]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\sin x\,dx\\ &=\frac{\pi^2}{4}-2\left\{\Big[x(-\cos x)\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx\right\}\\ &=\frac{\pi^2}{4}-2\Big[\sin x\Big]_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}-2.\end{aligned}$$

(13)　部分積分法より，

$$\int\frac{dx}{x^2+3}=\frac{x}{x^2+3}+\int\frac{2x^2}{(x^2+3)^2}\,dx=\frac{x}{x^2+3}+2\int\frac{(x^2+3)-3}{(x^2+3)^2}\,dx$$

$$= \frac{x}{x^2+3} + 2\int \frac{dx}{x^2+3} - 6\int \frac{dx}{(x^2+3)^2}.$$

よって

$$\int \frac{dx}{(x^2+3)^2} = \frac{1}{6}\left(\frac{x}{x^2+3} + \int \frac{dx}{x^2+3}\right) = \frac{1}{6}\left(\frac{x}{x^2+3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right).$$

ゆえに

$$\int_0^3 \frac{dx}{(x^2+3)^2} = \left[\frac{1}{6}\left(\frac{x}{x^2+3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)\right]_0^3 = \frac{\sqrt{3}}{54}\pi + \frac{1}{24}.$$

(14)　$\tan\frac{x}{2} = t$ とおくと，$\sin x = \frac{2t}{t^2+1}$，$dx = \frac{2}{t^2+1}\,dt$，$4+5\sin x = \frac{2(2t^2+5t+2)}{t^2+1}$．また，$x$ が 0 から $\frac{\pi}{2}$ まで動くとき，t は 0 から 1 まで動く．よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \int_0^1 \frac{dt}{2t^2+5t+2} = \frac{1}{3}\int_0^1 \left(\frac{2}{2t+1} - \frac{1}{t+2}\right)dt$$
$$= \frac{1}{3}\left[\log\left(t+\frac{1}{2}\right)\right]_0^1 - \frac{1}{3}\Big[\log(t+2)\Big]_0^1 = \frac{1}{3}\log 2.$$

(15)　$\cos x = t$ とおくと，$\sin x\,dx = -dt$，$\frac{1}{3+\tan^2 x} = \frac{\cos^2 x}{3\cos^2 x + \sin^2 x} = \frac{\cos^2 x}{2\cos^2 x + 1} = \frac{t^2}{2t^2+1}$．また，$x$ が 0 から $\frac{\pi}{3}$ まで動くとき，t は 1 から $\frac{1}{2}$ まで動く．よって

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{3+\tan^2 x}\,dx = -\int_1^{\frac{1}{2}} \frac{t^2}{2t^2+1}\,dt = -\int_0^{\frac{1}{2}} \frac{t^2+\frac{1}{2}-\frac{1}{2}}{2t^2+1}\,dt$$
$$-\frac{1}{2}\int_1^{\frac{1}{2}} dt + \frac{1}{4}\int_1^{\frac{1}{2}} \frac{dt}{t^2+\frac{1}{2}}$$
$$= -\frac{1}{2}\Big[t\Big]_1^{\frac{1}{2}} + \frac{1}{4}\Big[\sqrt{2}\tan^{-1}\sqrt{2}t\Big]_1^{\frac{1}{2}}$$
$$= \frac{1}{4} + \frac{\sqrt{2}}{4}\left(\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\sqrt{2}\right).$$

4.　(1)　$f(x) = \frac{1}{\sqrt{1+x}}$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{n+i}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{1+\frac{i}{n}}} = \int_0^1 \frac{dx}{\sqrt{1+x}}$$
$$= \Big[2\sqrt{1+x}\Big]_0^1 = 2\left(\sqrt{2}-1\right).$$

(2)　$f(x) = 2^x$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}2^{\frac{i}{n}} = \int_0^1 2^x\,dx = \left[\frac{2^x}{\log 2}\right]_0^1 = \frac{1}{\log 2}.$$

(3)　$f(x) = \sqrt{1-x^2}$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\frac{i^2}{n^2}} = \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_0^1 = \frac{1}{2}\sin^{-1} 1 = \frac{\pi}{4}.$$

(4)　$f(x) = \dfrac{1}{1+x^2}$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{1+\frac{i^2}{n^2}} = \int_0^1 \frac{dx}{1+x^2}$$
$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

(5)　$S_n := \left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}$ とおくと，

$$\log S_n = \frac{1}{n}\left\{\log\left(1+\frac{1^2}{n^2}\right) + \log\left(1+\frac{2^2}{n^2}\right) + \cdots + \log\left(1+\frac{n^2}{n^2}\right)\right\}$$
$$= \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{i^2}{n^2}\right).$$

ここで，$f(x) = \log(1+x^2)$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$\lim_{n\to\infty}\log S_n = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{i^2}{n^2}\right) = \int_0^1 \log(1+x^2)\,dx$$
$$= \left[x\log(1+x^2) - 2x + 2\tan^{-1} x\right]_0^1 = \log 2 - 2 + \frac{\pi}{2}.$$

ゆえに

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} e^{\log S_n} = e^{\log 2 - 2 + \frac{\pi}{2}} = 2e^{\frac{\pi}{2}-2}.$$

5.　(1)　加法定理より，

$$\sin mx\cos nx = \frac{1}{2}\big\{\sin(m+n)x + \sin(m-n)x\big\}.$$

- $m \neq n$ のとき:

$$\int_0^{2\pi}\sin mx\cos nx\,dx = \frac{1}{2}\int_0^{2\pi}\big\{\sin(m+n)x + \sin(m-n)x\big\}\,dx$$
$$= \frac{1}{2}\left[-\frac{1}{m+n}\cos(m+n)x - \frac{1}{m-n}\cos(m-n)x\right]_0^{2\pi} = 0.$$

- $m = n$ のとき:

$$\int_0^{2\pi}\sin mx\cos nx\,dx = \frac{1}{2}\int_0^{2\pi}\sin 2mx\,dx = \frac{1}{2}\left[-\frac{1}{2m}\cos 2mx\right]_0^{2\pi} = 0.$$

(2)　加法定理より，

$$\sin mx\sin nx = \frac{1}{2}\big\{-\cos(m+n)x + \cos(m-n)x\big\}$$
$$\cos mx\cos nx = \frac{1}{2}\big\{\cos(m+n)x + \cos(m-n)x\big\}.$$

- $m \neq n$ のとき:

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_0^{2\pi} \{-\cos(m+n)x + \cos(m-n)x\} \, dx$$
$$= \frac{1}{2} \left[-\frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right]_0^{2\pi} = 0.$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_0^{2\pi} \{\cos(m+n)x + \cos(m-n)x\} \, dx$$
$$= \frac{1}{2} \left[\frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right]_0^{2\pi} = 0.$$

- $m = n$ のとき

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_0^{2\pi} (-\cos 2mx + 1) \, dx = \frac{1}{2} \left[-\frac{1}{2m} \sin 2mx + x \right]_0^{2\pi} = \pi.$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_0^{2\pi} (\cos 2mx + 1) \, dx = \frac{1}{2} \left[\frac{1}{2m} \sin 2mx + x \right]_0^{2\pi} = \pi.$$

6. (1) 部分積分法より,

$$I(m,n) = \left[-\frac{1}{n+1} x^m (1-x)^{n+1} \right]_0^1 + \frac{m}{n+1} \int_0^1 x^{m-1} (1-x)^{n+1} \, dx$$
$$= \frac{m}{n+1} I(m-1, n+1).$$

(2) (1) を繰り返し用いると,

$$I(m,n) = \frac{m}{n+1} I(m-1, n+1) = \frac{m(m-1)}{(n+1)(n+2)} I(m-2, n+2)$$
$$= \frac{m(m-1)\cdots 2}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1)$$
$$= \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1).$$

ここで

$$I(1, n+m-1) = \int_0^1 x(1-x)^{n+m-1} \, dx$$
$$= \left[-\frac{1}{n+m} x(1-x)^{n+m} \right]_0^1 + \frac{1}{n+m} \int_0^1 (1-x)^{n+m} \, dx$$
$$= \frac{1}{n+m} \left[-\frac{1}{n+m+1} (1-x)^{n+m+1} \right]_0^1$$
$$= \frac{1}{(n+m)(n+m+1)}$$

なので,

$$I(m,n) = \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1)$$

$$= \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} \cdot \frac{1}{(n+m)(n+m+1)}$$
$$= \frac{m!}{(n+1)(n+2)\cdots(n+m+1)} = \frac{m!\,n!}{(m+n+1)!}.$$

7.　微分積分学の基本定理より，$M(x)$ は $(a,b]$ で微分可能で，

$$M'(x) = -\frac{1}{(x-a)^2}\int_a^x f(t)dt + \frac{f(x)}{x-a}.$$

任意に $x \in (a,b]$ を固定．積分の平均値の定理より，$c \in (a,x)$ が存在して，

$$\int_a^x f(t)dt = f(c)(x-a)$$

よって，

$$M'(x) = -\frac{1}{x-a}f(c) + \frac{f(x)}{x-a} = \frac{1}{x-a}\{f(x) - f(c)\}$$

を得る．上式において，

- $f(x)$ が狭義単調増加ならば $f(x) > f(c)$．よって $M'(x) > 0$.
- $f(x)$ が狭義単調減少ならば $f(x) < f(c)$．よって $M'(x) < 0$.

以上より，$M(x)$ は $(a,b]$ で狭義単調増加 (減少) である．

8.　$x = \pi - t$ とおくと，$dx = -dt$．また，x が 0 から π まで動くとき，t は π から 0 まで動く．よって

$$\int_0^\pi xf(\sin x)\,dx = \int_\pi^0 (\pi - t)f(\sin(\pi - t))\,(-dt) = \int_0^\pi (\pi - t)f(\sin t)\,dt$$
$$= \pi\int_0^\pi f(\sin t)\,dt - \int_0^\pi tf(\sin t)\,dt$$
$$= \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi xf(\sin x)\,dx.$$

ゆえに

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

9.　すべての $x \in [a,b]$ で $f(x) = 0$ のときは，シュワルツの不等式の左辺も右辺も 0 となり成り立つ．よって以下では，$[a,b]$ 内の少なくとも 1 点で $f(x)$ の値は 0 ではないと仮定する．このとき，定理 3.6 の (4) より

$$\int_a^b f(x)^2dx > 0$$

となる．

　さて，t は実数とする．このとき，任意の $x \in [a,b]$ に対して，

$$0 \leqq \{tf(x) + g(x)\}^2 = f(x)^2t^2 + 2f(x)g(x)t + g(x)^2$$

なので，両辺を x で a から b まで積分すると，

$$t^2\int_a^b f(x)^2dx + 2t\int_a^b f(x)g(x)dx + \int_a^b g(x)^2dx \geqq 0$$

となる．上式はすべての実数 t に対して成り立つので，

$$D/4 = \left(\int_a^b f(x)g(x)dx\right)^2 - \int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx \leqq 0$$

となり，シュワルツの不等式を得る．

10.　任意の $x \in [0,1]$ に対して $1 \leqq 1 + x^p \leqq 1 + x^2$ なので，

$$\frac{1}{\sqrt{1+x^2}} \leqq \frac{1}{\sqrt{1+x^p}} \leqq 1$$

となる．上の不等式に $x = \dfrac{1}{2}$ を代入すると，$0 < \left(\dfrac{1}{2}\right)^p < \left(\dfrac{1}{2}\right)^2$ より，

$$\frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}} < \frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^p}} < 1$$

である．よって，定理 3.6 の (4) より，

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} < \int_0^1 \frac{dx}{\sqrt{1+x^p}} < \int_0^1 dx$$

となるが，

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \left[\log\left|x + \sqrt{x^2+1}\right|\right]_0^1 = \log\left(1+\sqrt{2}\right)$$

なので，示すべき不等式を得る．

11.　(1)　$f(x) = \dfrac{x}{\sqrt{1-x^2}}$ は $[0,1)$ で連続なので，

$$\begin{aligned}\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx &= \lim_{\beta\to 1-0}\int_0^\beta \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{\beta\to 1-0}\left[-\sqrt{1-x^2}\right]_0^\beta \\ &= \lim_{\beta\to 1-0}\left(1-\sqrt{1-\beta^2}\right) = 1.\end{aligned}$$

(2)　$f(x) = \sqrt{\dfrac{1+x}{1-x}}$ は $[-1,1)$ で連続なので，

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\beta\to 1-0}\int_{-1}^\beta \sqrt{\frac{1+x}{1-x}}\,dx.$$

ここで，$\sqrt{\dfrac{1+x}{1-x}} = t$ とおくと，$x = \dfrac{t^2-1}{t^2+1}$，$dx = \dfrac{4t}{(t^2+1)^2}\,dt$．よって

$$\begin{aligned}\int \sqrt{\frac{1+x}{1-x}}\,dx &= \int \frac{4t^2}{(t^2+1)^2}\,dt = 4\int \frac{(t^2+1)-1}{(t^2+1)^2}\,dt \\ &= 4\int \frac{dt}{t^2+1} - 4\int \frac{dt}{(t^2+1)^2} \\ &= 4\tan^{-1} t - \frac{2t}{t^2+1} - 2\tan^{-1} t \\ &= 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2}.\end{aligned}$$

ゆえに

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx=\lim_{\beta\to1-0}\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}}-\sqrt{1-x^2}\right]_{-1}^{\beta}$$
$$=\lim_{\beta\to1-0}\left(2\tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}}-\sqrt{1-\beta^2}\right)=2\cdot\frac{\pi}{2}=\pi.$$

(3)　$f(x)=\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$ は $(0,\pi]$ で連続なので,

$$\int_0^{\pi}\left(\frac{\cos x}{x}-\frac{\sin x}{x^2}\right)dx=\lim_{\alpha\to+0}\int_{\alpha}^{\pi}\left(\frac{\cos x}{x}-\frac{\sin x}{x^2}\right)dx$$
$$=\lim_{\alpha\to+0}\int_{\alpha}^{\pi}\left(\frac{\sin x}{x}\right)'dx=\lim_{\alpha\to+0}\left[\frac{\sin x}{x}\right]_{\alpha}^{\pi}$$
$$=-\lim_{\alpha\to+0}\frac{\sin\alpha}{\alpha}=-1.$$

(4)　$f(x)=\dfrac{\log x}{\sqrt{x}}$ は $(0,1]$ で連続なので,

$$I:=\int_0^1\frac{\log x}{\sqrt{x}}\,dx=\lim_{\alpha\to+0}\int_{\alpha}^{1}\frac{\log x}{\sqrt{x}}\,dx.$$

ここで,

$$\int\frac{\log x}{\sqrt{x}}\,dx=2\sqrt{x}\log x-\int 2\sqrt{x}\cdot\frac{1}{x}\,dx=2\sqrt{x}\log x-2\int\frac{dx}{\sqrt{x}}=2\sqrt{x}\log x-4\sqrt{x}$$

である. よって,

$$I=\lim_{\alpha\to+0}\Big[2\sqrt{x}\log x-4\sqrt{x}\Big]_{\alpha}^{1}$$
$$=\lim_{\alpha\to+0}\left(-4-2\sqrt{\alpha}\log\alpha+4\sqrt{\alpha}\right)$$
$$=-4-2\lim_{\alpha\to+0}\sqrt{\alpha}\log\alpha$$

ロピタルの定理より,

$$\lim_{\alpha\to+0}\sqrt{\alpha}\log\alpha=\lim_{\alpha\to+0}\frac{\log\alpha}{\alpha^{-\frac{1}{2}}}=\lim_{\alpha\to+0}\frac{\frac{1}{\alpha}}{-\frac{1}{2}\alpha^{-\frac{3}{2}}}=-2\lim_{\alpha\to+0}\sqrt{\alpha}=0$$

なので, $I=-4$ となる.

(5)　$f(x)=\dfrac{1}{\sqrt{x(3-x)}}$ は $(0,3)$ で連続なので,

$$\int_0^3\frac{dx}{\sqrt{x(3-x)}}=\int_0^1\frac{dx}{\sqrt{x(3-x)}}+\int_1^3\frac{dx}{\sqrt{x(3-x)}}.$$

ここで, $\sqrt{\dfrac{x}{3-x}}=t\ \ (0<x<3)$ とおくと,

$$x=\frac{3t^2}{t^2+1},\quad dx=\frac{6t}{(t^2+1)^2}dt,\quad \frac{1}{\sqrt{x(3-x)}}=\frac{1}{x}\sqrt{\frac{x}{3-x}}=\frac{t(t^2+1)}{3t^2}.$$

よって

$$\int \frac{dx}{\sqrt{x(3-x)}} = \int \frac{t(t^2+1)}{3t^2} \cdot \frac{6t}{(t^2+1)^2}\,dt = 2\int \frac{dt}{t^2+1} = 2\tan^{-1} t = 2\tan^{-1}\sqrt{\frac{x}{3-x}}.$$

ゆえに

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x(3-x)}} &= \lim_{\alpha\to+0}\int_\alpha^1 \frac{dx}{\sqrt{x(3-x)}} = \lim_{\alpha\to+0}\left[2\tan^{-1}\sqrt{\frac{x}{3-x}}\right]_\alpha^1 \\ &= \lim_{\alpha\to+0} 2\left(\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\sqrt{\frac{\alpha}{3-\alpha}}\right) = 2\tan^{-1}\frac{1}{\sqrt{2}}.\end{aligned}$$

同様に

$$\begin{aligned}\int_1^\beta \frac{dx}{\sqrt{x(3-x)}} &= \lim_{\beta\to 3-0}\int_1^\beta \frac{dx}{\sqrt{x(3-x)}} = \lim_{\beta\to 3-0}\left[2\tan^{-1}\sqrt{\frac{x}{3-x}}\right]_1^\beta \\ &= \lim_{\beta\to 3-0} 2\left(\tan^{-1}\sqrt{\frac{\beta}{3-\beta}} - \tan^{-1}\frac{1}{\sqrt{2}}\right) \\ &= 2\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{2}}\right) = \pi - 2\tan^{-1}\frac{1}{\sqrt{2}}.\end{aligned}$$

以上より,

$$\int_0^3 \frac{dx}{\sqrt{x(3-x)}} = 2\tan^{-1}\frac{1}{\sqrt{2}} + \pi - 2\tan^{-1}\frac{1}{\sqrt{2}} = \pi.$$

(6) $f(x) = \dfrac{\cos x}{\sqrt{\sin x}}$ は $\left(0, \dfrac{\pi}{2}\right]$ で連続なので,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx &= \lim_{\alpha\to+0}\int_\alpha^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{\alpha\to+0}\left[2\sqrt{\sin x}\right]_\alpha^{\frac{\pi}{2}} \\ &= \lim_{\alpha\to+0} 2\left(1 - \sqrt{\sin\alpha}\right) = 2.\end{aligned}$$

(7) 部分積分を 2 回繰り返せば,

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx \\ &= -(x^2+2x+2)e^{-x}.\end{aligned}$$

$f(x) = x^2e^{-x}$ は $[0, \infty)$ で連続なので,

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= \lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx = \lim_{\beta\to\infty}\left[-(x^2+2x+2)e^{-x}\right]_0^\beta \\ &= \lim_{\beta\to\infty}\left\{2 - (\beta^2+2\beta+2)e^{-\beta}\right\} = 2.\end{aligned}$$

(8) $f(x) = \dfrac{1}{1+x^2}$ は $(-\infty, \infty)$ で連続なので,

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

である．ここで

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\beta\to\infty}\int_0^\beta \frac{dx}{1+x^2} = \lim_{\beta\to\infty}\left[\tan^{-1} x\right]_0^\beta = \lim_{\beta\to\infty}\tan^{-1}\beta = \frac{\pi}{2}$$

$$\int_{-\infty}^{0}\frac{dx}{1+x^2}=\lim_{\alpha\to-\infty}\int_{\alpha}^{0}\frac{dx}{1+x^2}=\lim_{\alpha\to-\infty}\Big[\tan^{-1}x\Big]_{\alpha}^{0}=-\lim_{\alpha\to-\infty}\tan^{-1}\alpha=\frac{\pi}{2}.$$

なので，$\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$

(9)　$f(x)=\dfrac{x^2}{(x^2+1)(x^2+4)}$ は $[0,\infty)$ で連続なので，

$$\int_{0}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}\,dx=\lim_{\beta\to\infty}\int_{0}^{\beta}\frac{x^2}{(x^2+1)(x^2+4)}\,dx.$$

ここで

$$\frac{x^2}{(x^2+1)(x^2+4)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}$$

とおくと，

$$x^2=(A+C)x^3+(B+D)x^2+(4A+C)x+4B+D.$$

両辺の係数を比較すると，$A+C=0$，$B+D=1$，$4A+C=0$，$4B+D=0$．よって，$A=C=0$，$B=-\dfrac{1}{3}$，$D=\dfrac{4}{3}$．ゆえに

$$\begin{aligned}\int_{0}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}\,dx&=\lim_{\beta\to\infty}\frac{1}{3}\int_{0}^{\beta}\left(-\frac{1}{x^2+1}+\frac{4}{x^2+4}\right)dx\\&=\lim_{\beta\to\infty}\frac{1}{3}\left[-\tan^{-1}x+2\tan^{-1}\frac{x}{2}\right]_{0}^{\beta}\\&=\lim_{\beta\to\infty}\frac{1}{3}\left(-\tan^{-1}\beta+2\tan^{-1}\frac{\beta}{2}\right)\\&=\frac{1}{3}\left(-\frac{\pi}{2}+2\cdot\frac{\pi}{2}\right)=\frac{\pi}{6}.\end{aligned}$$

参考．$x^2=X$ とおいて，与式を部分分数に分解すると，

$$\frac{x^2}{(x^2+1)(x^2+4)}=\frac{X}{(X+1)(X+4)}=\frac{1}{3}\left(\frac{4}{X+4}-\frac{1}{X+1}\right)=\frac{1}{3}\left(\frac{4}{x^2+4}-\frac{1}{x^2+1}\right).$$

(10)　$e^x=t$ とおくと，$e^x\,dx=dt$．よって

$$\begin{aligned}\int\frac{dx}{e^x+4e^{-x}+5}&=\int\frac{1}{t+\frac{4}{t}+5}\cdot\frac{dt}{t}=\int\frac{dt}{t^2+5t+4}\\&=\frac{1}{3}\int\left(\frac{1}{t+1}-\frac{1}{t+4}\right)dt=\frac{1}{3}\log\left|\frac{t+1}{t+4}\right|\\&=\frac{1}{3}\log\frac{e^x+1}{e^x+4}.\end{aligned}$$

ここで，$f(x)=\dfrac{1}{e^x+4e^{-x}+5}$ は $[0,\infty)$ で連続なので，

$$\begin{aligned}\int_{0}^{\infty}\frac{dx}{e^x+4e^{-x}+5}&=\lim_{\beta\to\infty}\int_{0}^{\beta}\frac{dx}{e^x+4e^{-x}+5}=\lim_{\beta\to\infty}\left[\frac{1}{3}\log\frac{e^x+1}{e^x+4}\right]_{0}^{\beta}\\&=\frac{1}{3}\lim_{\beta\to\infty}\left(\log\frac{e^\beta+1}{e^\beta+4}-\log\frac{2}{5}\right)=-\frac{1}{3}\log\frac{2}{5}.\end{aligned}$$

(11)　$x+\sqrt{x^2+1}=t$ とおくと，$x=\dfrac{t^2-1}{2t}$，$dx=\dfrac{t^2+1}{2t^2}\,dt$．よって

$$\begin{aligned}\int\frac{dx}{(x+\sqrt{x^2+1})^2}&=\int\frac{1}{t^2}\cdot\frac{t^2+1}{2t}\,dt=\frac{1}{2}\int\frac{t^2+1}{t^4}\,dt\\&=\frac{1}{2}\int\left(t^{-2}+t^{-4}\right)dt=\frac{1}{2}\left(-t^{-1}-\frac{1}{3}t^{-3}\right)\\&=-\frac{1}{2}\cdot\frac{1}{x+\sqrt{x^2+1}}-\frac{1}{3}\cdot\frac{1}{(x+\sqrt{x^2+1})^3}.\end{aligned}$$

ここで，$f(x)=\dfrac{1}{(x+\sqrt{x^2+1})^2}$ は $[0,\infty)$ で連続なので，

$$\begin{aligned}\int_0^{\infty}\frac{dx}{(x+\sqrt{x^2+1})^2}&=\lim_{\beta\to\infty}\int_0^{\beta}\frac{dx}{(x+\sqrt{x^2+1})^2}\\&=\lim_{\beta\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{x+\sqrt{x^2+1}}-\frac{1}{3}\cdot\frac{1}{(x+\sqrt{x^2+1})^3}\right]_0^{\beta}\\&=\lim_{\beta\to\infty}\left\{-\frac{1}{2}\cdot\frac{1}{\beta+\sqrt{\beta^2+1}}-\frac{1}{3}\cdot\frac{1}{(\beta+\sqrt{\beta^2+1})^3}+\frac{1}{2}+\frac{1}{6}\right\}\\&=\frac{2}{3}.\end{aligned}$$

(12)　$f(x)=\left(x-\sqrt{x^2-1}\right)^4$ は $[1,\infty)$ で連続なので，

$$\int_1^{\infty}\left(x-\sqrt{x^2-1}\right)^4dx=\lim_{\beta\to\infty}\int_1^{\beta}\left(x-\sqrt{x^2-1}\right)^4dx.$$

ここで，$x-\sqrt{x^2-1}=t$ とおくと，$x=\dfrac{t^2+1}{2t}$，$dx=\dfrac{t^2-1}{2t^2}\,dt$．よって

$$\begin{aligned}\int\left(x-\sqrt{x^2-1}\right)^4dx&=\int t^4\cdot\frac{t^2-1}{2t^2}\,dt=\frac{1}{2}\int(t^4-t^2)\,dt=\frac{1}{2}\left(\frac{t^5}{5}-\frac{t^3}{3}\right)\\&=\frac{1}{10}\left(x-\sqrt{x^2-1}\right)^5-\frac{1}{6}\left(x-\sqrt{x^2-1}\right)^3.\end{aligned}$$

ゆえに

$$\begin{aligned}\int_1^{\infty}\left(x-\sqrt{x^2-1}\right)^4dx&=\lim_{\beta\to\infty}\left[\frac{1}{10}\left(x-\sqrt{x^2-1}\right)^5-\frac{1}{6}\left(x-\sqrt{x^2-1}\right)^3\right]_1^{\beta}\\&=\lim_{\beta\to\infty}\left\{\frac{1}{10}\left(\beta-\sqrt{\beta^2-1}\right)^5-\frac{1}{6}\left(\beta-\sqrt{\beta^2-1}\right)^3+\frac{1}{15}\right\}\\&=\lim_{\beta\to\infty}\left\{\frac{1}{10}\cdot\frac{1}{(\beta+\sqrt{\beta^2-1})^5}-\frac{1}{6}\cdot\frac{1}{(\beta+\sqrt{\beta^2-1})^3}\right\}+\frac{1}{15}\\&=\frac{1}{15}.\end{aligned}$$

12.　(1)　$f(x)=x^{p-1}\log x$ は $(0,1]$ で連続なので，

$$\int_0^1x^{p-1}\log x\,dx=\lim_{\alpha\to+0}\int_{\alpha}^1x^{p-1}\log x\,dx$$

である．ここで，部分積分法より，

$$\int x^{p-1}\log x\,dx = \frac{x^p}{p}\log x - \int \frac{x^p}{p}\frac{1}{x}\,dx = \frac{x^p}{p}\log x - \frac{1}{p}\int x^{p-1}\,dx$$
$$= \frac{x^p}{p}\log x - \frac{1}{p^2}x^p.$$

よって，

$$I = \int_0^1 x^{p-1}\log x\,dx = \lim_{\alpha\to+0}\left[\frac{1}{p}x^p\log x - \frac{1}{p^2}x^p\right]_\alpha^1$$
$$= \lim_{\alpha\to+0}\left(-\frac{1}{p^2} - \frac{1}{p}\alpha^p\log\alpha + \frac{1}{p^2}\alpha^p\right) = -\frac{1}{p^2} - \frac{1}{p}\lim_{\alpha\to+0}\alpha^p\log\alpha.$$

ロピタルの定理より，

$$\lim_{\alpha\to+0}\alpha^p\log\alpha = \lim_{\alpha\to+0}\frac{\log\alpha}{\alpha^{-p}} = \lim_{\alpha\to+0}\frac{\frac{1}{\alpha}}{-p\alpha^{-p-1}} = -\frac{1}{p}\lim_{\alpha\to+0}\alpha^p = 0$$

なので，$I = -\dfrac{1}{p^2}$ となる．

(2)　$f(x) = \dfrac{1}{\sqrt{(b-x)(x-a)}}$ は (a,b) で連続なので，

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\int_\alpha^\beta \frac{dx}{\sqrt{(b-x)(x-a)}}\right)$$

である．ここで，$\sqrt{\dfrac{b-x}{x-a}} = t$ とおくと，

$$x = \frac{at^2+b}{t^2+1},\quad x-a = \frac{b-a}{t^2+1},\quad dx = -\frac{2(b-a)t}{(t^2+1)^2}\,dt$$

なので，

$$\int \frac{dx}{\sqrt{(b-x)(x-a)}} = \int \frac{dx}{(x-a)\sqrt{\dfrac{b-x}{x-a}}} = -\int \frac{t^2+1}{(b-a)t}\cdot\frac{2(b-a)t}{(t^2+1)^2}\,dt$$
$$= -2\int\frac{dt}{t^2+1} = -2\tan^{-1}t = -2\tan^{-1}\sqrt{\frac{b-x}{x-a}}.$$

を得る．よって，

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\left[-2\tan^{-1}\sqrt{\frac{b-x}{x-a}}\right]_\alpha^\beta\right)$$
$$= \lim_{\alpha\to a-0}2\tan^{-1}\sqrt{\frac{b-\alpha}{\alpha-a}} - \lim_{\beta\to b-0}2\tan^{-1}\sqrt{\frac{b-\beta}{\beta-a}} = \pi.$$

別解.　$f(x) = \dfrac{1}{\sqrt{(b-x)(x-a)}}$ は (a,b) で連続なので，

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \int_a^b \frac{dx}{\sqrt{\left(\dfrac{b-a}{2}\right)^2 - \left(x-\dfrac{a+b}{2}\right)^2}}$$

$$= \lim_{\alpha \to a-0} \left(\lim_{\beta \to b-0} \int_\alpha^\beta \frac{dx}{\sqrt{\left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}} \right)$$

$$= \lim_{\alpha \to a-0} \left(\lim_{\beta \to b-0} \left[\sin^{-1} \frac{2x-(a+b)}{b-a} \right]_\alpha^\beta \right)$$

$$= \lim_{\beta \to b-0} \sin^{-1} \frac{2\beta-(a+b)}{b-a} - \lim_{\alpha \to a-0} \sin^{-1} \frac{2\alpha-(a+b)}{b-a}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

(3)　$f(\theta) = 1 - 2r\cos\theta + r^2$ とおくと，

$$f(\theta) = \sin^2\theta + \cos^2\theta - 2r\cos\theta + r^2 = \sin^2\theta + (r - \cos\theta)^2$$

なので，$f(\theta) = 0 \iff \sin\theta = 0$ かつ $r = \cos\theta$．このとき，$r = \cos\theta = \pm 1$ となり，$-1 < r < 1$ に反する．よって，

$$\frac{1}{f(\theta)} = \frac{1}{1 - 2r\cos\theta + r^2}$$

は $[0, 2\pi]$ で連続なので，積分可能である．また，$f(\pi - \theta) = f(\pi + \theta)$ なので，$1/f(\theta)$ は $\theta = \pi$ に関して対称．ゆえに，

$$I = \int_0^{2\pi} \frac{d\theta}{f(\theta)} = 2\int_0^\pi \frac{d\theta}{f(\theta)} = 2\int_0^\pi \frac{d\theta}{1 - 2r\cos\theta + r^2}$$

となる．ここで，$\tan\dfrac{\theta}{2} = t$ $(-\pi < \theta < \pi)$ とおくと，$d\theta = \dfrac{2dt}{1+t^2}$，$\cos\theta = \dfrac{1-t^2}{1+t^2}$ なので，

$$\int \frac{d\theta}{1 - 2r\cos\theta + r^2} = 2\int \frac{dt}{(1+r)^2t^2 + (1-r)^2} = \frac{2}{(1+r)^2} \int \frac{dt}{t^2 + \left(\frac{1-r}{1+r}\right)^2}$$

$$= \frac{2}{1-r^2} \tan^{-1} \frac{(1+r)t}{1-r} = \frac{2}{1-r^2} \tan^{-1} \frac{(1+r)\tan\frac{\theta}{2}}{1-r}$$

となる．よって，$0 < \beta < \pi$ を満たす任意の β に対して，θ が 0 から β まで動くとき，t は 0 から $\tan\dfrac{\beta}{2}$ まで動くので，置換積分法より，

$$\int_0^\beta \frac{d\theta}{1 - 2p\cos\theta + p^2} = \left[\frac{2}{1-r^2} \tan^{-1} \frac{(1+r)\tan\frac{\theta}{2}}{1-r} \right]_0^\beta = \frac{2}{1-r^2} \tan^{-1} \frac{(1+r)\tan\frac{\beta}{2}}{1-r}$$

を得る．ゆえに，微分積分学の基本定理より，

$$I = 2\lim_{\beta \to \pi-0} \int_0^\beta \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{4}{1-r^2} \lim_{\beta \to \pi-0} \tan^{-1} \frac{(1+r)\tan\frac{\beta}{2}}{1-r}$$

$$= \frac{4}{1-r^2} \cdot \frac{\pi}{2} = \frac{2\pi}{1-r^2}.$$

13.　(1)　$0 < x < \dfrac{\pi}{2}$ のとき，$0 < \sin x < 1$．よって，$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$．ゆえに，定理 3.6 の (4) より，(1) の不等式が成り立つ．

(2)　例 3.2.8 を用いて，(1) の不等式を書き直すと，

$$\frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot \cdots \cdot (2n+1)} < \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot \cdots \cdot (2n)} \cdot \frac{\pi}{2} < \frac{2 \cdot 4 \cdot \cdots \cdot (2n-2)}{1 \cdot 3 \cdot \cdots \cdot (2n-1)}$$

となる．上式のすべての項に

$$\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}$$

を掛けると，(2) の不等式を得る．

(3)　(2) の不等式を変形すると，

$$\frac{2n}{2n+1}<\frac{\pi}{\dfrac{1}{n}\left\{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right\}^2}<1$$

となる．よって，$n\to\infty$ とすると，$\dfrac{2n}{2n+1}\to 1$ より，

$$\pi=\lim_{n\to\infty}\frac{1}{n}\left\{\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right\}^2$$

を得る．ゆえに，

$$\sqrt{\pi}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{\{2\cdot 4\cdot\cdots\cdot(2n)\}^2}{(2n)!}=\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!}.$$

14.　以下では，求める面積を S で表す．

(1)　2 曲線 $y=\sin x$ と $y=\cos 2x$ の $0\leqq x\leqq\pi$ での交点の x 座標は $x=\dfrac{\pi}{6},\dfrac{5\pi}{6}$．よって，図形の対称性より，

$$\begin{aligned}S&=2\left\{\int_0^{\frac{\pi}{6}}(\cos 2x-\sin x)\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\sin x-\cos 2x)\,dx\right\}\\&=2\left\{\left[\frac{\sin 2x}{2}+\cos x\right]_0^{\frac{\pi}{6}}+\left[-\cos x-\frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}\\&=2\left\{\left(\frac{3}{4}\sqrt{3}-1\right)+\frac{3}{4}\sqrt{3}\right)=3\sqrt{3}-2.\end{aligned}$$

(2)　2 曲線 $y=x^2$ と $\sqrt{x}+\sqrt{y}=2$ の $0\leqq x\leqq 4$ での交点の x 座標は $x=1$．よって

$$\begin{aligned}S&=\int_0^1\left\{\left(2-\sqrt{x}\right)^2-x^2\right\}dx=\int_0^1\left(4-4\sqrt{x}+x-x^2\right)dx\\&=\left[4x-\frac{8}{3}x^{\frac{3}{2}}+\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=4-\frac{8}{3}+\frac{1}{2}-\frac{1}{3}=\frac{3}{2}.\end{aligned}$$

(3)　2 曲線 $y^2=4x$ と $x^2=4y$ の交点の x 座標は $x=0,4$．よって

$$S=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{x^3}{12}\right]_0^4=\frac{4}{3}4^{\frac{3}{2}}-\frac{1}{12}4^3=\frac{16}{3}.$$

(4)　媒介変数表示式が表す曲線と x 軸との交点の x 座標を求めるために，$y=2-t-t^2=0$ とおくと，$t=-2,1$．これを $x=2t+1$ に代入して，$x=-3,3$ を得る．$x=2t+1$，$y=2-t-t^2$ なので，$dx=2dt$ である．また，x が -3 から 3 まで動くとき，t は -2 から 1 まで動くので，

$$S=\int_{-3}^3 y\,dx=\int_{-2}^1\left(2-t-t^2\right)2\,dt=2\left[2t-\frac{t^2}{2}-\frac{t^3}{3}\right]_{-2}^1=9.$$

(5)　$x=t^2$，$y=t^3$ なので，$dx=2tdt$．また，x が 0 から 1 まで動くとき，t も 0 から 1 まで動く．よって，図形の対称性より

$$S=2\int_0^1 y\,dx=2\int_0^1 t^3\cdot 2t\,dt=4\int_0^1 t^4\,dt=4\left[\frac{t^5}{5}\right]_0^1=\frac{4}{5}.$$

(6)　曲線と x 軸との交点は $(\pm 1,0)$，y 軸との交点は $(0,\pm 1)$ である．$x=\cos^3 t$，$y=\sin^3 t$ なので，$dx=3\cos^3 t\,(-\sin t)\,dt=-3\sin t\cos^2 t\,dt$．また，$x$ が 0 から 1 まで動くとき，t は $\frac{\pi}{2}$ から 0 まで動く．よって，図形の対称性より，

$$\begin{aligned}S&=4\int_0^1 y\,dx=4\int_{\frac{\pi}{2}}^0 \sin^3 t(-3\sin t\cos^3 t)\,dt=12\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,dt\\&=12\left\{\int_0^{\frac{\pi}{2}}\sin^4 t\,dt-\int_0^{\frac{\pi}{2}}\sin^6 t\,dt\right\}\\&=12\left\{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\}\quad(\text{例 3.2.8 の公式})\\&=\frac{3\pi}{8}\end{aligned}$$

15.　以下では，求める面積を S で表す．

(1)　$2\sin^2\theta=1-\cos 2\theta$ なので，

$$S=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(2\sin\theta^2)\,d\theta=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(1-\cos 2\theta)\,d\theta=\left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}=\frac{\pi}{2}+1.$$

(2)　$\tan^2\theta=\dfrac{\sin^2\theta}{\cos^2\theta}=\dfrac{1-\cos^2\theta}{\cos^2\theta}=\dfrac{1}{\cos^2\theta}-1$ なので，

$$S=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^2\theta\,d\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{\cos^2\theta}-1\right)d\theta=\frac{1}{2}\Big[\tan\theta-\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{1}{\sqrt{3}}-\frac{\pi}{12}.$$

(3)　$(1+\cos\theta)^2=1+2\cos\theta+\cos^2\theta=1+2\cos\theta+\dfrac{1+\cos 2\theta}{2}$ なので，

$$\begin{aligned}S&=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos\theta)^2\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta\\&=\frac{1}{2}\left[\theta+2\sin\theta+\frac{\theta+\frac{\sin 2\theta}{2}}{2}\right]_0^{\frac{\pi}{2}}=\frac{3}{8}\pi+1.\end{aligned}$$

16.　以下では，求める曲線の長さを L で表す．

(1)　$x=at^2$，$y=2at$ なので，$x'=2at$，$y'=2a$．よって，$(x')^2+(y')^2=4a^2t^2+4a^2=4a^2(1+t^2)$．ゆえに

$$\begin{aligned}L&=\int_0^1\sqrt{4a^2(1+t^2)}\,dt=2a\int_0^1\sqrt{t^2+1}\,dt\\&=2a\left[\frac{1}{2}\left(t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right)\right]_0^1\quad(\text{例 3.1.6 (1) の公式})\\&=a\left\{\sqrt{2}+\log\left(1+\sqrt{2}\right)\right\}.\end{aligned}$$

(2)　$x=t\cos\dfrac{1}{t}$，$y=t\sin\dfrac{1}{t}$ なので，

$$x'=\cos\frac{1}{t}+t\left(-\sin\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)=\cos\frac{1}{t}+\frac{1}{t}\sin\frac{1}{t},$$

$$y' = \sin\frac{1}{t} + t\left(\cos\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

よって

$$(x')^2 + (y')^2 = \left(\cos\frac{1}{t} + \frac{1}{t}\sin\frac{1}{t}\right)^2 + \left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right)^2 = \frac{t^2+1}{t^2}$$

より，

$$\sqrt{(x')^2 + (y')^2} = \frac{\sqrt{t^2+1}}{t} \quad (t > 0)$$

となる．ゆえに，

$$L = \int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt$$

である．$x = \sqrt{t^2+1}$ とおくと，$dx = \dfrac{t}{\sqrt{t^2+1}}\,dt$．また，$t$ が 1 から 2 まで動くとき，x は $\sqrt{2}$ から $\sqrt{5}$ まで動く．よって，

$$\frac{\sqrt{t^2+1}}{t}dt = \frac{t^2+1}{t^2}\cdot\frac{t}{\sqrt{t^2+1}}dt = \frac{x^2}{x^2-1}dx$$

に注意すると，

$$\begin{aligned} L &= \int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt = \int_{\sqrt{2}}^{\sqrt{5}} \frac{x^2}{x^2-1}\,dx = \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{x^2-1}\right)dx \\ &= \left[x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_{\sqrt{2}}^{\sqrt{5}} \\ &= \sqrt{5} - \sqrt{2} + \frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{\sqrt{5}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right\}. \end{aligned}$$

ここで，

$$\log\frac{\sqrt{5}-1}{\sqrt{5}+1} = \log\frac{\left(\sqrt{5}-1\right)^2}{5-1} = \log\frac{\left(\sqrt{5}-1\right)^2}{4} = 2\log\left(\sqrt{5}-1\right) - 2\log 2$$

$$\log\frac{\sqrt{2}-1}{\sqrt{2}+1} = \log\frac{\left(\sqrt{2}-1\right)^2}{2-1} = 2\log\left(\sqrt{2}-1\right)$$

なので，

$$L = \sqrt{5} - \sqrt{2} + \log\left(\sqrt{5}-1\right) - \log 2 - \log\left(\sqrt{2}-1\right) = \sqrt{5} - \sqrt{2} + \log\frac{\sqrt{5}-1}{2\left(\sqrt{2}-1\right)}.$$

(3)　$r = a\theta$ なので，$r' = a$．よって，$\sqrt{r^2 + (r')^2} = \sqrt{a^2\theta^2 + a^2} = a\sqrt{1+\theta^2}$．ゆえに

$$\begin{aligned} L &= a\int_0^{2\pi} \sqrt{\theta^2+1}\,d\theta \\ &= a\left[\frac{1}{2}\left(\theta\sqrt{\theta^2+1} + \log\left|\theta + \sqrt{\theta^2+1}\right|\right)\right]_0^{2\pi} \quad (\text{例 3.1.6 (1) の公式}) \\ &= a\left\{\pi\sqrt{4\pi^2+1} + \frac{1}{2}\log\left(2\pi + \sqrt{4\pi^2+1}\right)\right\} \end{aligned}$$

(4)　$r = a\sin\theta$ なので，$r' = a\cos\theta$．よって，$\sqrt{r^2 + (r')^2} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$．ゆえに

$$L = \int_0^{\pi} a\,d\theta = \pi a.$$

17. 以下では，求める立体の体積を V で表す.

(1) xy-平面と平行な平面 $z=z$ での切り口は楕円 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1-\dfrac{z^4}{c^4}$ なので，その面積 $S(z)$ は

$$S(z)=\pi ab\left(1-\frac{z^4}{c^4}\right)$$

である．よって，図形の対称性より，

$$V=2\int_0^c S(z)\,dz=2\pi ab\int_0^c\left(1-\frac{z^4}{c^4}\right)dz=2\pi ab\left[z-\frac{z^5}{5c^4}\right]_0^c=\frac{8}{5}\pi abc.$$

(2) yz-平面と平行な平面 $x=x$ での切り口は，1 辺が $2\sqrt{a^2-x^2}$ の正方形なので，その面積 $S(x)$ は，$S(x)=4\left(a^2-x^2\right)$．よって，図形の対称性より，

$$V=2\int_0^a S(x)\,dx=8\int_0^a\left(a^2-x^2\right)dx=8\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{16}{3}a^3.$$

18. 以下では，求める回転体の体積を V で表す.

(1) 曲線 $y=\sqrt{x}$ と直線 $y=\dfrac{x}{2}$ の交点の x 座標は $x=0,4$ である．よって

$$\begin{aligned}V&=\pi\int_0^4\left(\sqrt{x}\right)^2dx-\pi\int_0^4\left(\frac{x}{2}\right)^2dx\\&=\pi\int_0^4 x\,dx-\frac{\pi}{4}\int_0^4 x^2dx=\pi\left[\frac{x^2}{2}\right]_0^4-\frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^4=\frac{8}{3}\pi.\end{aligned}$$

(2) $\tan^2 x=\dfrac{\sin^2 x}{\cos^2 x}=\dfrac{1-\cos^2 x}{\cos^2 x}=\dfrac{1}{\cos^2 x}-1$ なので，

$$V=\pi\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=\pi\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)dx=\pi\Big[\tan x-x\Big]_0^{\frac{\pi}{4}}=\pi-\frac{\pi^2}{4}.$$

(3) $V=\pi\displaystyle\int_{\sqrt{2}}^3\left(x^2-2\right)dx=\pi\left[\frac{x^3}{3}-2x\right]_{\sqrt{2}}^3=\pi\left(3+\frac{4\sqrt{2}}{3}\right).$

(4) $x=t^2$，$y=t^3$ なので，$dx=2tdt$，$y^2=t^6$．また，x が 0 から 1 まで動くとき，t も 0 から 1 まで動く．よって

$$V=\pi\int_0^1 y^2dx=\pi\int_0^1 t^6\cdot 2t\,dt=2\pi\int_0^1 t^7dt=2\pi\left[\frac{t^8}{8}\right]_0^1=\frac{\pi}{4}.$$

(5) $x=t-1$，$y=4t-t^2$ なので，$dx=dt$，$y^2=\left(4t-t^2\right)^2=16t^2-8t^3+t^4$．また，$x$ が 0 から 2 まで動くとき，t は 1 から 3 まで動く．よって，

$$V=\pi\int_0^2 y^2dx=\pi\int_1^3\left(16t^2-8t^3+t^4\right)dt=\pi\left[\frac{16}{3}t^3-2t^4+\frac{t^5}{5}\right]_1^3=\frac{406}{15}\pi.$$