3　積分

演習問題 3

1.　(1)　$\displaystyle\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx=\int\left(x-2+\frac{1}{x}\right)dx=\frac{x^2}{2}-2x+\log|x|$

(2)　$\displaystyle\int(1-x)\sqrt[3]{x}\,dx=\int\left(x^{\frac{1}{3}}-x^{\frac{4}{3}}\right)dx=\frac{3}{4}x^{\frac{4}{3}}-\frac{3}{7}x^{\frac{7}{3}}=\frac{3}{28}x(7-4x)\sqrt[3]{x}$

(3)　$\displaystyle\frac{2}{1-x^2}=\frac{1}{1-x}+\frac{1}{1+x}$ より,

$$\begin{aligned}\int\left(\frac{2}{1-x^2}+\frac{3}{\sqrt{1-x^2}}\right)dx&=\int\left(\frac{1}{1-x}+\frac{1}{1+x}+\frac{3}{\sqrt{1-x^2}}\right)dx\\&=-\log|1-x|+\log|1+x|+3\sin^{-1}x\\&=\log\left|\frac{1+x}{1-x}\right|+3\sin^{-1}x\end{aligned}$$

(4)　$\sqrt[3]{x+1}=t$ とおくと，$x=t^3-1$，$dx=3t^2dt$．よって

$$\begin{aligned}\int x\sqrt[3]{x+1}\,dx&=\int(t^3-1)t\cdot3t^2dt=3\int(t^6-t^3)\,dt=3\left(\frac{t^7}{7}-\frac{t^4}{4}\right)\\&=\frac{3}{28}t^3(4t^3-7)t=\frac{3}{28}(x+1)(4x-3)\sqrt[3]{x+1}.\end{aligned}$$

別解.　$$\begin{aligned}\int x\sqrt[3]{x+1}\,dx&=\int\{(x+1)-1\}\sqrt[3]{x+1}\,dx=\int\left\{(x+1)^{\frac{4}{3}}-(x+1)^{\frac{1}{3}}\right\}dx\\&=\frac{3}{7}(x+1)^{\frac{7}{3}}-\frac{3}{4}(x+1)^{\frac{4}{3}}=\frac{3}{28}(x+1)(4x-3)\sqrt[3]{x+1}.\end{aligned}$$

(5)　$\displaystyle\frac{2x+3}{(x+2)(x+1)^2}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$ とおくと,

$$\begin{aligned}2x+3&=A(x+1)^2+B(x+2)(x+1)+C(x+2)\\&=(A+B)x^2+(2A+3B+C)x+(A+2B+2C).\end{aligned}$$

両辺の係数を比較すると，$A+B=0$，$2A+3B+C=2$，$A+2B+2C=3$．これを解いて，$A=-1$，$B=1$，$C=1$．よって

$$\frac{2x+3}{(x+2)(x+1)^2}=-\frac{1}{x+2}+\frac{1}{x+1}+\frac{1}{(x+1)^2}$$

と部分分数分解できる．よって

$$\begin{aligned}\int\frac{2x+3}{(x+2)(x+1)^2}\,dx&=\int\left\{-\frac{1}{x+2}+\frac{1}{x+1}+\frac{1}{(x+1)^2}\right\}dx\\&=-\log|x+2|+\log|x+1|-\frac{1}{x+1}\\&=\log\left|\frac{x+1}{x+2}\right|-\frac{1}{x+1}.\end{aligned}$$

(6)　$\displaystyle\frac{x^5-x^2+1}{x^3-1}=x^2+\frac{1}{x^3-1}=x^2+\frac{1}{3}\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}\right)$ なので,

$$\int\frac{x^5-x^2+1}{x^3-1}\,dx$$

$$= \int \left\{ x^2 + \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} \right) \right\} dx$$

$$= \int \left\{ x^2 + \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} dx$$

$$= \frac{x^3}{3} + \frac{1}{3}\log|x-1| - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$$

$$= \frac{x^3}{3} + \frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

(7) $\dfrac{1}{x^4+x^2+1} = \dfrac{1}{(x^2+1)^2-x^2} = \dfrac{1}{(x^2-x+1)(x^2+x+1)}$ なので，

$$\frac{1}{x^4+x^2+1} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1}$$

とおくと，

$$\begin{aligned} 1 &= (Ax+B)(x^2+x+1) + (Cx+D)(x^2-x+1) \\ &= (A+C)x^3 + (A+B-C+D)x^2 + (A+B+C-D)x + B + D. \end{aligned}$$

両辺の係数を比較すると，$A+C=0$，$A+B-C+D=0$，$A+B+C-D=0$，$B+D=1$．
これを解いて，$A=-\dfrac{1}{2}$，$B=C=D=\dfrac{1}{2}$．よって

$$\frac{1}{x^4+x^2+1} = \frac{1}{2}\left(-\frac{x-1}{x^2-x+1} + \frac{x+1}{x^2+x+1} \right).$$

ゆえに

$$\begin{aligned} \int \frac{dx}{x^4+x^2+1} &= \frac{1}{2}\int \left(-\frac{x-1}{x^2-x+1} + \frac{x+1}{x^2+x+1} \right) dx \\ &= \frac{1}{4}\int \left\{ -\frac{(2x-1)-1}{x^2-x+1} + \frac{(2x+1)+1}{x^2+x+1} \right\} dx \\ &= \frac{1}{4}\int \left\{ -\frac{2x-1}{x^2-x+1} + \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right. \\ &\qquad \left. + \frac{2x+1}{x^2+x+1} + \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} dx \\ &= \frac{1}{4}\left\{ -\log(x^2-x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right) \right. \\ &\qquad \left. + \log(x^2+x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \right\} \\ &= \frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\left(\tan^{-1}\frac{2x-1}{\sqrt{3}} + \tan^{-1}\frac{2x+1}{\sqrt{3}} \right). \end{aligned}$$

(8) $x=e^t$ とくと，$t=\log x$，$dx=e^t dt$．よって

$$\int x^2\log x\,dx = \int \left(e^t\right)^2 t\cdot e^t dt = \int te^{3t}dt = t\left(\frac{1}{3}e^{3t}\right) - \int \frac{1}{3}e^{3t}dt$$

$$= \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} = \frac{x^3}{9}(3\log x - 1).$$

別解. $\displaystyle\int x^2 \log x\,dx = \frac{x^3}{3}\log x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2\,dx$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} = \frac{x^3}{9}(3\log x - 1).$$

(9) $\displaystyle\int e^x \sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$ より,

$$\int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x).$$

(10) $x = \sin t$ $\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$ とおくと，$dx = \cos t\,dt$，$x^2\sqrt{1-x^2} = \sin^2 t\sqrt{1-\sin^2 t} = \sin^2 t\cos t$．よって

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos t}{\sin^2 t\cos t}\,dt = \int \frac{dt}{\sin^2 t} = -\cot t$$

$$= -\frac{\cos t}{\sin t} = -\frac{\sqrt{1-\sin^2 t}}{\sin t} = -\frac{\sqrt{1-x^2}}{x}.$$

(11) $\log x = t$ とおくと，$x = e^t$，$dx = e^t dt$．よって

$$\int \frac{dx}{x(\log x)^3} = \int \frac{e^t}{e^t\cdot t^3}\,dt = \int t^{-3}dt = -\frac{1}{2}t^{-2} = -\frac{1}{2(\log x)^2}.$$

別解. $\displaystyle\int \frac{dx}{x(\log x)^3} = \int (\log x)'(\log x)^{-3}\,dx = -\frac{1}{2}(\log x)^{-2} = -\frac{1}{2(\log x)^2}.$

(12) $\sqrt{e^{2x}+1} = t$ とおくと，$e^{2x}+1 = t^2$，$e^{2x}dx = tdt$．よって，$dx = \dfrac{t}{t^2-1}\,dt$．ゆえに

$$\int \sqrt{e^{2x}+1}\,dx = \int \frac{t^2}{t^2-1}\,dt = \int \left(1 + \frac{1}{t^2-1}\right)dt$$

$$= \int \left\{1 + \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)\right\}dt$$

$$= t + \frac{1}{2}(\log|t-1| - \log|t+1|)$$

$$= t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$= \sqrt{e^{2x}+1} + \frac{1}{2}\log\frac{\sqrt{e^{2x}+1}-1}{\sqrt{e^{2x}+1}+1}$$

$$= \sqrt{e^{2x}+1} + \frac{1}{2}\log\frac{(\sqrt{e^{2x}+1}-1)^2}{e^{2x}}$$

$$= \sqrt{e^{2x}+1} + \log(\sqrt{e^{2x}+1}-1) - x.$$

(13) $\sin^{-1}x = t$ とおくと，$x = \sin t$ $\left(-\frac{\pi}{2} \leqq t \leqq \frac{\pi}{2}\right)$，$dx = \cos t\,dt$．よって

$$\int (\sin^{-1}x)^2\,dx = \int t^2\cos t\,dt = t^2\sin t - 2\int t\sin t\,dt$$

$$
\begin{aligned}
&= t^2\sin t + 2t\cos t - 2\int \cos t\,dt \\
&= t^2\sin t + 2t\sqrt{1-\sin^2 t} - 2\sin t \\
&= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x.
\end{aligned}
$$

(14)　$\sqrt{x-1} = t$ とおくと，$x = t^2+1$，$dx = 2t\,dt$．よって

$$
\begin{aligned}
\int \frac{dx}{x+\sqrt{x-1}} &= \int \frac{2t}{t^2+t+1}\,dt = \int \frac{(2t+1)-1}{t^2+t+1}\,dt \\
&= \int \frac{2t+1}{t^2+t+1}\,dt - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right) \\
&= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}} \\
&= \log\left(x+\sqrt{x-1}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x-1}+1}{\sqrt{3}}.
\end{aligned}
$$

(15)　$\log x = t$ とおくと，$\dfrac{dx}{x} = dt$．よって

$$
\int \frac{\sin(\log x)}{x}\,dx = \int \sin t\,dt = -\cos t = -\cos(\log x).
$$

別解.　$\displaystyle\int \frac{\sin(\log x)}{x}\,dx = \int (\log x)'\sin(\log x)\,dx = -\cos(\log x).$

(16)　$\sqrt{\dfrac{x}{x-1}} = t$ とおくと，$x = \dfrac{t^2}{t^2-1}$，$dx = -\dfrac{2t}{(t^2-1)^2}\,dt$．よって

$$
\begin{aligned}
\int \frac{1}{x}\sqrt{\frac{x}{x-1}}\,dx &= -\int \frac{t^2-1}{t^2}\cdot\frac{2t^2}{(t^2-1)^2}\,dt = -2\int \frac{dt}{t^2-1} \\
&= -\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt = \log|t+1| - \log|t-1| \\
&= \log\left|\frac{t+1}{t-1}\right| = \log\left|\frac{\sqrt{\frac{x}{x-1}}+1}{\sqrt{\frac{x}{x-1}}-1}\right| \\
&= \log\left|\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}\right| = 2\log\left(\sqrt{x}+\sqrt{x-1}\right).
\end{aligned}
$$

(17)　$x = \sqrt{2}\tan t\ \left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$ とおくと，

$$
dx = \frac{\sqrt{2}}{\cos^2 t}\,dt,\quad \left(x^2+2\right)^{\frac{5}{2}} = \left(2\tan^2 t+2\right)^{\frac{5}{2}} = 4\sqrt{2}\left(1+\tan^2 t\right)^{\frac{5}{2}} = \frac{4\sqrt{2}}{\cos^5 t}.
$$

よって

$$
\int \frac{x^2}{(x^2+2)^{\frac{5}{2}}}\,dx = \int 2\tan^2 t\cdot\frac{\cos^5 t}{4\sqrt{2}}\cdot\frac{\sqrt{2}}{\cos^2 t}\,dt = \frac{1}{2}\int \tan^2 t\cos^3 t\,dt
$$

$$= \frac{1}{2}\int \sin^2 t\cos t\,dt = \frac{1}{6}\sin^3 t$$
$$= \frac{1}{6}\left(\frac{\tan^2 t}{\tan^2 t+1}\right)^{\frac{3}{2}} = \frac{1}{6}\left(\frac{x^2}{x^2+2}\right)^{\frac{3}{2}}.$$

(18)　$x=\sin t$ $\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$ とおくと，$dx=\cos t\,dt$．よって

$$\int \frac{dx}{(1-x^2)^{\frac{5}{2}}} = \int \frac{\cos t}{(1-\sin^2 t)^{\frac{5}{2}}}\,dt = \int \frac{\cos t}{\cos^5 t}\,dt = \int \frac{dt}{\cos^4 t}.$$

そこで，$\tan t = y$ とおくと，$\frac{dt}{\cos^2 t}=dy$，$\frac{1}{\cos^4 t}=\frac{1+\tan^2 t}{\cos^2 t}$．よって

$$\int \frac{dt}{\cos^4 t} = \int \left(1+y^2\right)dy = y+\frac{y^3}{3} = \tan t + \frac{\tan^3 t}{3}.$$

さて，$\cos^2 t = 1-\sin^2 t = 1-x^2$，$\cos t \geqq 0$ なので，$\cos t=\sqrt{1-x^2}$．よって，$\tan t = \frac{x}{\sqrt{1-x^2}}$．
ゆえに

$$\int \frac{dx}{(1-x^2)^{\frac{5}{2}}} = \tan t + \frac{\tan^3 t}{3} = \frac{x(3-2x^2)}{3\,(1-x^2)^{\frac{3}{2}}}.$$

(19)　$\sqrt{x^2-x+1}=t-x$ とおくと

$$x=\frac{t^2-1}{2t-1},\quad dx=\frac{2(t^2-t+1)}{(2t-1)^2}\,dt,\quad \sqrt{x^2-x+1}=\frac{t^2-t+1}{2t-1}$$

より

$$\int \frac{dx}{x\sqrt{x^2-x+1}} = \int \frac{2t-1}{t^2-1}\cdot\frac{2t-1}{t^2-t+1}\cdot\frac{2(t^2-t+1)}{(2t-1)^2}\,dt = 2\int\frac{dt}{t^2-1}$$
$$= \int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt = \log|t-1|-\log|t+1|$$
$$= \log\left|\frac{t-1}{t+1}\right| = \log\left|\frac{\sqrt{x^2-x+1}+x-1}{\sqrt{x^2-x+1}+x+1}\right|.$$

(20)　$\sqrt{2+x-x^2}=\sqrt{(2-x)(x+1)}=(2-x)\sqrt{\frac{x+1}{2-x}}$ より，$\sqrt{\frac{x+1}{2-x}}=t$ とおくと，

$$x=\frac{2t^2+1}{t^2+1},\quad \sqrt{2+x-x^2}=\frac{3t}{t^2+1},\quad dx=\frac{6t}{\left(t^2+1\right)^2}dt.$$

よって

$$\int\frac{dx}{\sqrt{2+x-x^2}} = \int\frac{t^2+1}{3t}\cdot\frac{6t}{\left(t^2+1\right)^2}\,dt = 2\int\frac{dt}{t^2+1} = 2\tan^{-1}t = 2\tan^{-1}\sqrt{\frac{x+1}{2-x}}.$$

別解．$\sqrt{2+x-x^2}=\sqrt{(2-x)(x+1)}=(x+1)\sqrt{\frac{2-x}{x+1}}$ より，$\sqrt{\frac{2-x}{x+1}}=t$ とおくと，

$$x=-\frac{t^2-1}{t^2+1},\quad \sqrt{2+x-x^2}=\frac{2t}{t^2+1},\quad dx=-\frac{4t}{(t^2+1)^2}\,dt.$$

よって

$$\int \frac{dx}{\sqrt{2+x-x^2}} = \int \frac{t^2+1}{2t} \cdot \frac{-4t}{(t^2+1)^2}\,dt = -2\int \frac{dt}{t^2+1}$$
$$= -2\tan^{-1} t = -2\tan^{-1}\sqrt{\frac{2-x}{x+1}}.$$

別解. $\displaystyle\int \frac{dx}{\sqrt{2+x-x^2}} = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}} = \sin^{-1}\frac{2}{3}\left(x-\frac{1}{2}\right) = \sin^{-1}\frac{2x-1}{3}.$

(21) $\tan\dfrac{x}{2} = t$ とおくと,

$$dx = \frac{2}{1+t^2}\,dt, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad 1+2\cos x = \frac{3-t^2}{1+t^2}.$$

よって

$$\int \frac{dx}{1+2\cos x} = \int \frac{1+t^2}{3-t^2}\cdot\frac{2}{1+t^2}\,dt = -2\int \frac{dt}{t^2-3}$$
$$= \frac{1}{\sqrt{3}}\int\left(\frac{1}{t+\sqrt{3}} - \frac{1}{t-\sqrt{3}}\right)dt$$
$$= \frac{1}{\sqrt{3}}\left(\log|t+\sqrt{3}| - \log|t-\sqrt{3}|\right)$$
$$= \frac{1}{\sqrt{3}}\log\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right| = \frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{x}{2}+\sqrt{3}}{\tan\frac{x}{2}-\sqrt{3}}\right|.$$

2. $n \neq 0, 1$ とする．部分積分法を用いて計算すると,

$$I_n = \int \sin^n x\,dx = \int \sin^{n-1}x\sin x\,dx = -\int \sin^{n-1}x(\cos x)'\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

となる．よって，漸化式

$$I_n = \frac{1}{n}\left\{-\sin^{n-1}x\cos x + (n-1)I_{n-2}\right\} \quad (n \neq 0, 1) \qquad (*)$$

を得る．

$n = 1$ のときは,

$$I_1 = \int \sin x\,dx = -\cos x.$$

一方，(*) で $n = 1$ とおくと,

$$I_1 = \frac{1}{1}\left\{-(\sin x)^0\cos x + (1-1)I_{-1}\right\} = -\cos x.$$

よって，(*) は $n=1$ のときも成り立つ．以上より，漸化式 (*) は 0 以外のすべての整数 n で成り立つ．

次に，(*) に $n=2,4$ を代入すると，

$$I_2 = -\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0,\quad I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2.$$

さらに，I_0 の定義より，

$$I_0 = \int (\sin x)^0\,dx = \int dx = x.$$

以上より，

$$\begin{aligned}\int \sin^4 x\,dx = I_4 &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\cos x + \frac{x}{2}\right)\\ &= \frac{1}{8}\left\{-2\sin^3 x\cos x - 3\sin x\cos x + 3x\right\}.\end{aligned}$$

また，(*) に $n=-2$ を代入すると，

$$I_{-2} = \frac{1}{2}\cdot\frac{\cos x}{\sin^3 x} + \frac{3}{2}I_{-4}.$$

さらに，I_{-2} の定義より，

$$I_{-2} = \int \frac{1}{\sin^2 x}dx = -\frac{1}{\tan x}.$$

よって

$$\begin{aligned}\int \frac{dx}{\sin^4 x} = I_{-4} = \frac{2}{3}\left(-\frac{1}{2}\frac{\cos x}{\sin^3 x} + I_{-2}\right) &= -\frac{\cos x}{3\sin^3 x} - \frac{2}{3\tan x}\\ &= -\frac{\cos x}{3\sin^3 x}(1+2\sin^2 x).\end{aligned}$$

3. (1) $\displaystyle\int_1^{e^2} \frac{\log x + 1}{x}\,dx = \left[\frac{1}{2}(\log x+1)^2\right]_1^{e^2} = \frac{1}{2}(9-1) = 4.$

(2) $\sqrt{x+1} = t$ とおくと，$x+1=t^2$，$dx = 2tdt$．また，x が 0 から 1 まで動くとき，t は 1 から $\sqrt{2}$ まで動く．よって

$$\begin{aligned}\int_0^1 x\sqrt{x+1}\,dx &= \int_1^{\sqrt{2}} \left(t^2-1\right)\cdot t\cdot(2t\,dt) = 2\int_1^{\sqrt{2}}\left(t^4-t^2\right)dt\\ &= 2\left[\frac{t^5}{5}-\frac{t^3}{3}\right]_1^{\sqrt{2}} = \frac{2}{5}\left(4\sqrt{2}-1\right) - \frac{2}{3}\left(2\sqrt{2}-1\right)\\ &= \frac{4}{15}\left(1+\sqrt{2}\right).\end{aligned}$$

別解.
$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int\{(x+1)-1\}\sqrt{x+1}\,dx = \int\left\{(x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}}\right\}dx\\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}.\end{aligned}$$

よって

$$\int_0^1 x\sqrt{x+1}\,dx = \left[\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^1 = \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}$$

$$= \frac{4}{15}\left(1+\sqrt{2}\right).$$

(3)　$\sqrt{x+1}=t$ とおくと，$x+1=t^2$，$dx=2tdt$．また，x が 0 から 1 まで動くとき，t は 1 から $\sqrt{2}$ まで動く．よって

$$\begin{aligned}\int_0^1 \frac{x^2}{\sqrt{x+1}}\,dx &= \int_1^{\sqrt{2}} \frac{\left(t^2-1\right)^2}{t}\cdot 2t\,dt = 2\int_1^{\sqrt{2}}\left(t^4-2t^2+1\right)dt \\ &= 2\left[\frac{t^5}{5}-\frac{2}{3}t^3+t\right]_1^{\sqrt{2}} \\ &= \frac{2}{5}\left(4\sqrt{2}-1\right)-\frac{4}{3}\left(2\sqrt{2}-1\right)+2\left(\sqrt{2}-1\right) \\ &= \frac{2}{15}\left(7\sqrt{2}-8\right).\end{aligned}$$

別解.
$$\begin{aligned}\int \frac{x^2}{\sqrt{x+1}}\,dx &= \int \frac{(x+1)^2-2(x+1)+1}{\sqrt{x+1}}\,dx \\ &= \int \left\{(x+1)^{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}+(x+1)^{-\frac{1}{2}}\right\}dx \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2(x+1)^{\frac{1}{2}}.\end{aligned}$$

これより

$$\begin{aligned}\int_0^1 \frac{x^2}{\sqrt{x+1}}\,dx &= \left[\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2(x+1)^{\frac{1}{2}}\right]_0^1 \\ &= \frac{8\sqrt{2}}{5}-\frac{8\sqrt{2}}{3}+2\sqrt{2}-\frac{2}{5}+\frac{4}{3}-2 \\ &= \frac{14\sqrt{2}}{15}-\frac{16}{15}=\frac{2}{15}\left(7\sqrt{2}-8\right).\end{aligned}$$

(4)　$\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_0^2 (4-x^2)'(4-x^2)^{\frac{1}{2}}\,dx = -\frac{1}{2}\left[\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2 = \frac{8}{3}.$

(5)　$$\begin{aligned}\int_0^1 \frac{dx}{x^2+x+1} &= \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1 \\ &= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{9}\pi.\end{aligned}$$

(6)　$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}} &= \frac{1}{2}\int_1^2\left(\sqrt{x+1}+\sqrt{x-1}\right)dx \\ &= \frac{1}{2}\left[\frac{2}{3}(x+1)^{\frac{3}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 = \sqrt{3}-\frac{2}{3}\sqrt{2}+\frac{1}{3}.\end{aligned}$$

(7)　$x=\tan\theta$ とおくと，$dx=\dfrac{d\theta}{\cos^2\theta}$．また，$x$ が 0 から 1 まで動くとき，θ は 0 から $\dfrac{\pi}{4}$ まで動く．よって

$$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(\tan^2\theta+1)^2}\cdot\frac{d\theta}{\cos^2\theta} = \int_0^{\frac{\pi}{4}}\sin^2\theta\cos^2\theta\,d\theta.$$

ゆえに

$$\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}\cdot\frac{1-\cos 4\theta}{2} = \frac{1}{8}(1-\cos 4\theta)$$

より,

$$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} (1-\cos 4\theta)\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{32}.$$

別解. $\dfrac{x^2}{(x^2+1)^3} = \dfrac{(x^2+1)-1}{(x^2+1)^3} = \dfrac{1}{(x^2+1)^2} - \dfrac{1}{(x^2+1)^3}$ と部分分数分解できるので,

$$\int \frac{x^2}{(x^2+1)^3}\,dx = \int \frac{dx}{(x^2+1)^2} - \int \frac{dx}{(x^2+1)^3}.$$

ここで,

$$\begin{aligned}\int \frac{dx}{x^2+1} &= \frac{x}{x^2+1} + \int \frac{2x^2}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + 2\int \frac{(x^2+1)-1}{(x^2+1)^2}\,dx \\ &= \frac{x}{x^2+1} + 2\int \frac{dx}{x^2+1} - 2\int \frac{dx}{(x^2+1)^2}\end{aligned}$$

より

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right) = \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1} x\right).$$

また

$$\begin{aligned}\int \frac{dx}{(x^2+1)^2} &= \frac{x}{(x^2+1)^2} + \int \frac{4x^2}{(x^2+1)^3}\,dx = \frac{x}{(x^2+1)^2} + 4\int \frac{(x^2+1)-1}{(x^2+1)^3}\,dx \\ &= \frac{x}{(x^2+1)^2} + 4\int \frac{dx}{(x^2+1)^2} - 4\int \frac{dx}{(x^2+1)^3}\end{aligned}$$

より

$$\begin{aligned}\int \frac{dx}{(x^2+1)^3} &= \frac{1}{4}\left\{\frac{x}{(x^2+1)^2} + 3\int \frac{dx}{(x^2+1)^2}\right\} \\ &= \frac{x}{4(x^2+1)^2} + \frac{3}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right).\end{aligned}$$

ゆえに

$$\begin{aligned}\int \frac{x^2}{(x^2+1)^3}\,dx &= \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2} - \frac{3}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) \\ &= \frac{1}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2}.\end{aligned}$$

よって

$$\begin{aligned}\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx &= \left[\frac{1}{8}\left(\frac{x}{x^2+1} + \tan^{-1} x\right) - \frac{x}{4(x^2+1)^2}\right]_0^1 \\ &= \frac{1}{8}\tan^{-1} 1 = \frac{1}{8}\cdot\frac{\pi}{4} = \frac{\pi}{32}.\end{aligned}$$

(8) $\sqrt{x^2+x+1} = t - x$ とおくと,

$$x = \frac{t^2-1}{2t+1}, \quad \sqrt{x^2+x+1} = \frac{t^2+t+1}{2t+1}, \quad dx = \frac{2(t^2+t+1)}{(2t+1)^2}\,dt.$$

また，x が 1 から 2 まで動くとき，t は $1+\sqrt{3}$ から $2+\sqrt{7}$ まで動く．よって

$$\begin{aligned}\int_1^2 \frac{dx}{x\sqrt{x^2+x+1}} &= \int_{1+\sqrt{3}}^{2+\sqrt{7}} \frac{1}{\dfrac{t^2-1}{2t+1}\cdot\dfrac{t^2+t+1}{2t+1}}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt\\ &= \int_{1+\sqrt{3}}^{2+\sqrt{7}} \frac{2}{t^2-1}\,dt = \left[\log\left|\frac{t-1}{t+1}\right|\right]_{1+\sqrt{3}}^{2+\sqrt{7}}\\ &= \log\frac{1+\sqrt{7}}{3+\sqrt{7}} - \log\frac{\sqrt{3}}{2+\sqrt{3}}.\end{aligned}$$

(9)　$\log x = t$ とおくと，$\dfrac{dx}{x} = dt$，$x = e^t$．また，x が 1 から e まで動くとき，t は 0 から 1 まで動く．そこで，$I := \displaystyle\int_1^e \cos(\log x)\,dx$ とおくと，

$$\begin{aligned}I &= \int_0^1 e^t\cos t\,dt = \Big[e^t\sin t\Big]_0^1 - \int_0^1 e^t\sin t\,dt\\ &= e\sin 1 - \left\{\Big[e^t(-\cos t)\Big]_0^1 - \int_0^1 e^t(-\cos t)\,dt\right\}\\ &= e(\sin 1+\cos 1) - 1 - I.\end{aligned}$$

ゆえに，$I = \dfrac{e}{2}(\sin 1+\cos 1) - \dfrac{1}{2}$.

別解.　部分積分すると

$$\begin{aligned}\int\cos(\log x)\,dx &= x\cos(\log x) + \int\sin(\log x)\,dx\\ &= x\cos(\log x) + x\sin(\log x) - \int\cos(\log x)\,dx.\end{aligned}$$

よって

$$\int\cos(\log x)\,dx = \frac{x}{2}\big\{\cos(\log x)+\sin(\log x)\big\}.$$

ゆえに

$$\int_1^e\cos(\log x)\,dx = \left[\frac{x}{2}\left\{\cos(\log x)+\sin(\log x)\right\}\right]_1^e = \frac{e}{2}(\cos 1+\sin 1) - \frac{1}{2}.$$

(10)
$$\begin{aligned}\int x^2\tan^{-1}x\,dx &= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{x^2+1}\,dx\\ &= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx\\ &= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log(x^2+1).\end{aligned}$$

よって

$$\int_0^1 x^2\tan^{-1}x\,dx = \left[\frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log(x^2+1)\right]_0^1 = \frac{\pi}{12} + \frac{1}{6}(\log 2 - 1)$$

(11)　$\tan\dfrac{x}{2} = t$ とおくと，$\sin x = \dfrac{2t}{1+t^2}$，$dx = \dfrac{2dt}{1+t^2}$．また，$x$ が 0 から $\dfrac{\pi}{3}$ まで動くとき，t

は0から$\dfrac{1}{\sqrt{3}}$まで動く．よって

$$\int_0^{\frac{\pi}{3}} \frac{dx}{1+\sin x} = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = 2\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{(t+1)^2}$$

$$= 2\left[-\frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}} = \sqrt{3}-1$$

別解. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1+\sin x} = \int_0^{\frac{\pi}{3}} \frac{1-\sin x}{\cos^2 x}\,dx = \left[\tan x - \frac{1}{\cos x}\right]_0^{\frac{\pi}{3}} = \sqrt{3}-1.$

(12)　部分積分法より，

$$\int_0^{\frac{\pi}{2}} x^2\cos x\,dx = \Big[x^2\sin x\Big]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} x\sin x\,dx$$

$$= \frac{\pi^2}{4} - 2\left\{\Big[x(-\cos x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx\right\}$$

$$= \frac{\pi^2}{4} - 2\Big[\sin x\Big]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - 2.$$

(13)　部分積分法より，

$$\int \frac{dx}{x^2+3} = \frac{x}{x^2+3} + \int \frac{2x^2}{(x^2+3)^2}\,dx = \frac{x}{x^2+3} + 2\int \frac{(x^2+3)-3}{(x^2+3)^2}\,dx$$

$$= \frac{x}{x^2+3} + 2\int \frac{dx}{x^2+3} - 6\int \frac{dx}{(x^2+3)^2}.$$

よって

$$\int \frac{dx}{(x^2+3)^2} = \frac{1}{6}\left(\frac{x}{x^2+3} + \int \frac{dx}{x^2+3}\right) = \frac{1}{6}\left(\frac{x}{x^2+3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right).$$

ゆえに

$$\int_0^3 \frac{dx}{(x^2+3)^2} = \left[\frac{1}{6}\left(\frac{x}{x^2+3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)\right]_0^3 = \frac{\sqrt{3}}{54}\pi + \frac{1}{24}.$$

(14)　$\tan\dfrac{x}{2} = t$とおくと，$\sin x = \dfrac{2t}{t^2+1}$，$dx = \dfrac{2}{t^2+1}\,dt$，$4+5\sin x = \dfrac{2(2t^2+5t+2)}{t^2+1}$．また，$x$が$0$から$\dfrac{\pi}{2}$まで動くとき，$t$は$0$から$1$まで動く．よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \int_0^1 \frac{dt}{2t^2+5t+2} = \frac{1}{3}\int_0^1 \left(\frac{2}{2t+1} - \frac{1}{t+2}\right)dt$$

$$= \frac{1}{3}\left[\log\left(t+\frac{1}{2}\right)\right]_0^1 - \frac{1}{3}\Big[\log(t+2)\Big]_0^1 = \frac{1}{3}\log 2.$$

(15)　$\cos x = t$とおくと，$\sin x\,dx = -dt$，$\dfrac{1}{3+\tan^2 x} = \dfrac{\cos^2 x}{3\cos^2 x+\sin^2 x} = \dfrac{\cos^2 x}{2\cos^2 x+1} = \dfrac{t^2}{2t^2+1}$．また，$x$が$0$から$\dfrac{\pi}{3}$まで動くとき，$t$は$1$から$\dfrac{1}{2}$まで動く．よって

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{3+\tan^2 x}\,dx = -\int_1^{\frac{1}{2}} \frac{t^2}{2t^2+1}\,dt = -\int_0^{\frac{1}{2}} \frac{t^2+\frac{1}{2}-\frac{1}{2}}{2t^2+1}\,dt$$

$$
\begin{aligned}
&-\frac{1}{2}\int_1^{\frac{1}{2}} dt + \frac{1}{4}\int_1^{\frac{1}{2}} \frac{dt}{t^2+\frac{1}{2}} \\
&= -\frac{1}{2}\Big[\,t\,\Big]_1^{\frac{1}{2}} + \frac{1}{4}\Big[\sqrt{2}\tan^{-1}\sqrt{2}t\Big]_1^{\frac{1}{2}} \\
&= \frac{1}{4} + \frac{\sqrt{2}}{4}\left(\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\sqrt{2}\right).
\end{aligned}
$$

4. (1) $f(x)=\dfrac{1}{\sqrt{1+x}}$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{n+i}} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{1+\frac{i}{n}}} = \int_0^1\frac{dx}{\sqrt{1+x}} \\
&= \Big[\,2\sqrt{1+x}\,\Big]_0^1 = 2\left(\sqrt{2}-1\right).
\end{aligned}
$$

(2) $f(x)=2^x$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}2^{\frac{i}{n}} = \int_0^1 2^x\,dx = \left[\frac{2^x}{\log 2}\right]_0^1 = \frac{1}{\log 2}.
$$

(3) $f(x)=\dfrac{1}{\sqrt{1-x^2}}$ は $[0,1)$ で連続で，

$$
\int_0^1\frac{dx}{\sqrt{1-x^2}} = \lim_{\beta\to 1-0}\int_0^{\beta}\frac{dx}{\sqrt{1-x^2}} = \lim_{\beta\to 1-0}\Big[\sin^{-1}x\Big]_0^{\beta} = \lim_{\beta\to 1-0}\sin^{-1}\beta = \frac{\pi}{2}
$$

なので，広義積分は収束する．よって，区分求積公式より，

$$
\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-i^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\sqrt{1-\frac{i^2}{n^2}}} = \int_0^1\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.
$$

(4) $f(x)=\dfrac{1}{1+x^2}$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{n}{n^2+i^2} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{1+\frac{i^2}{n^2}} = \int_0^1\frac{dx}{1+x^2} \\
&= \Big[\tan^{-1}x\Big]_0^1 = \tan^{-1}1 = \frac{\pi}{4}.
\end{aligned}
$$

(5) $S_n := \left\{\left(1+\dfrac{1^2}{n^2}\right)\left(1+\dfrac{2^2}{n^2}\right)\cdots\left(1+\dfrac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}$ とおくと，

$$
\begin{aligned}
\log S_n &= \frac{1}{n}\left\{\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots+\log\left(1+\frac{n^2}{n^2}\right)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{i^2}{n^2}\right).
\end{aligned}
$$

ここで，$f(x)=\log(1+x^2)$ は $[0,1]$ で連続なので積分可能．よって，区分求積公式より，

$$
\lim_{n\to\infty}\log S_n = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{i^2}{n^2}\right) = \int_0^1\log(1+x^2)\,dx
$$

$$= \left[x\log(1+x^2) - 2x + 2\tan^{-1} x \right]_0^1 = \log 2 - 2 + \frac{\pi}{2}.$$

ゆえに

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} e^{\log S_n} = e^{\log 2 - 2 + \frac{\pi}{2}} = 2e^{\frac{\pi}{2}-2}.$$

5. (1) 加法定理より,

$$\sin mx \cos nx = \frac{1}{2}\left\{\sin(m+n)x + \sin(m-n)x\right\}.$$

- $m \neq n$ のとき:

$$\int_0^{2\pi} \sin mx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} \left\{\sin(m+n)x + \sin(m-n)x\right\} dx$$

$$= \frac{1}{2}\left[-\frac{1}{m+n}\cos(m+n)x - \frac{1}{m-n}\cos(m-n)x \right]_0^{2\pi} = 0.$$

- $m = n$ のとき:

$$\int_0^{2\pi} \sin mx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx \, dx = \frac{1}{2}\left[-\frac{1}{2m}\cos 2mx \right]_0^{2\pi} = 0.$$

(2) 加法定理より,

$$\sin mx \sin nx = \frac{1}{2}\left\{-\cos(m+n)x + \cos(m-n)x\right\}$$

$$\cos mx \cos nx = \frac{1}{2}\left\{\cos(m+n)x + \cos(m-n)x\right\}.$$

- $m \neq n$ のとき:

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_0^{2\pi} \left\{-\cos(m+n)x + \cos(m-n)x\right\} dx$$

$$= \frac{1}{2}\left[-\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x \right]_0^{2\pi} = 0.$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} \left\{\cos(m+n)x + \cos(m-n)x\right\} dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x \right]_0^{2\pi} = 0.$$

- $m = n$ のとき

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_0^{2\pi} (-\cos 2mx + 1) \, dx = \frac{1}{2}\left[-\frac{1}{2m}\sin 2mx + x \right]_0^{2\pi} = \pi.$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} (\cos 2mx + 1) \, dx = \frac{1}{2}\left[\frac{1}{2m}\sin 2mx + x \right]_0^{2\pi} = \pi.$$

6. (1) 部分積分法より,

$$I(m,n) = \left[-\frac{1}{n+1}x^m(1-x)^{n+1} \right]_0^1 + \frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1} \, dx$$

$$= \frac{m}{n+1} I(m-1, n+1).$$

(2)　(1) を繰り返し用いると，

$$\begin{aligned}
I(m,n) &= \frac{m}{n+1} I(m-1, n+1) = \frac{m(m-1)}{(n+1)(n+2)} I(m-2, n+2) \\
&= \frac{m(m-1)\cdots 2}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1) \\
&= \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1).
\end{aligned}$$

ここで

$$\begin{aligned}
I(1, n+m-1) &= \int_0^1 x(1-x)^{n+m-1}\, dx \\
&= \left[-\frac{1}{n+m} x(1-x)^{n+m} \right]_0^1 + \frac{1}{n+m} \int_0^1 (1-x)^{n+m}\, dx \\
&= \frac{1}{n+m} \left[-\frac{1}{n+m+1} (1-x)^{n+m+1} \right]_0^1 \\
&= \frac{1}{(n+m)(n+m+1)}
\end{aligned}$$

なので，

$$\begin{aligned}
I(m,n) &= \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} I(1, n+m-1) \\
&= \frac{m!}{(n+1)(n+2)\cdots(n+m-1)} \cdot \frac{1}{(n+m)(n+m+1)} \\
&= \frac{m!}{(n+1)(n+2)\cdots(n+m+1)} = \frac{m!\, n!}{(m+n+1)!}.
\end{aligned}$$

7.　微分積分学の基本定理より，$M(x)$ は $(a, b]$ で微分可能で，

$$M'(x) = -\frac{1}{(x-a)^2} \int_a^x f(t)dt + \frac{f(x)}{x-a}.$$

任意に $x \in (a, b]$ を固定．積分の平均値の定理より，$c \in (a, x)$ が存在して，

$$\int_a^x f(t)dt = f(c)(x-a)$$

よって，

$$M'(x) = -\frac{1}{x-a} f(c) + \frac{f(x)}{x-a} = \frac{1}{x-a} \left\{ f(x) - f(c) \right\}$$

を得る．上式において，

- $f(x)$ が狭義単調増加ならば $f(x) > f(c)$．よって $M'(x) > 0$.
- $f(x)$ が狭義単調減少ならば $f(x) < f(c)$．よって $M'(x) < 0$.

以上より，$M(x)$ は $(a,b]$ で狭義単調増加 (減少) である．

8.　$x=\pi-t$ とおくと，$dx=-dt$．また，x が 0 から π まで動くとき，t は π から 0 まで動く．よって

$$\begin{aligned}\int_0^\pi xf(\sin x)\,dx &= \int_\pi^0 (\pi-t)f(\sin(\pi-t))\,(-dt) = \int_0^\pi (\pi-t)f(\sin t)\,dt \\ &= \pi\int_0^\pi f(\sin t)\,dt - \int_0^\pi tf(\sin t)\,dt \\ &= \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi xf(\sin x)\,dx.\end{aligned}$$

ゆえに

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

9.　すべての $x\in[a,b]$ で $f(x)=0$ のときは，シュワルツの不等式の左辺も右辺も 0 となり成り立つ．よって以下では，$[a,b]$ 内の少なくとも 1 点で $f(x)$ の値は 0 ではないと仮定する．このとき，定理 3.6 の (4) より

$$\int_a^b f(x)^2dx > 0$$

となる．

さて，t は実数とする．このとき，任意の $x\in[a,b]$ に対して，

$$0 \leqq \{tf(x)+g(x)\}^2 = f(x)^2t^2 + 2f(x)g(x)t + g(x)^2$$

なので，両辺を x で a から b まで積分すると，

$$t^2\int_a^b f(x)^2dx + 2t\int_a^b f(x)g(x)dx + \int_a^b g(x)^2dx \geqq 0$$

となる．上式はすべての実数 t に対して成り立つので，

$$D/4 = \left(\int_a^b f(x)g(x)dx\right)^2 - \int_a^b f(x)^2dx\cdot\int_a^b g(x)^2dx \leqq 0$$

となり，シュワルツの不等式を得る．

10.　任意の $x\in[0,1]$ に対して $1\leqq 1+x^p\leqq 1+x^2$ なので，

$$\frac{1}{\sqrt{1+x^2}} \leqq \frac{1}{\sqrt{1+x^p}} \leqq 1$$

となる．上の不等式に $x=\dfrac{1}{2}$ を代入すると，$0<\left(\dfrac{1}{2}\right)^p<\left(\dfrac{1}{2}\right)^2$ より，

$$\frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}} < \frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^p}} < 1$$

である．よって，定理 3.6 の (4) より，

$$\int_0^1\frac{dx}{\sqrt{1+x^2}} < \int_0^1\frac{dx}{\sqrt{1+x^p}} < \int_0^1 dx$$

となるが，

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \left[\log\left|x+\sqrt{x^2+1}\right|\right]_0^1 = \log\left(1+\sqrt{2}\right)$$

なので，示すべき不等式を得る．

11.　(1)　$f(x) = \dfrac{x}{\sqrt{1-x^2}}$ は $[0,1)$ で連続なので，

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{\beta\to 1-0}\int_0^\beta \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{\beta\to 1-0}\left[-\sqrt{1-x^2}\right]_0^\beta$$

$$= \lim_{\beta\to 1-0}\left(1-\sqrt{1-\beta^2}\right) = 1.$$

(2)　$f(x) = \sqrt{\dfrac{1+x}{1-x}}$ は $[-1,1)$ で連続なので，

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\beta\to 1-0}\int_{-1}^\beta \sqrt{\frac{1+x}{1-x}}\,dx.$$

ここで，$\sqrt{\dfrac{1+x}{1-x}} = t$ とおくと，$x = \dfrac{t^2-1}{t^2+1}$，$dx = \dfrac{4t}{(t^2+1)^2}\,dt$．よって

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{4t^2}{(t^2+1)^2}\,dt = 4\int \frac{(t^2+1)-1}{(t^2+1)^2}\,dt$$

$$= 4\int \frac{dt}{t^2+1} - 4\int \frac{dt}{(t^2+1)^2}$$

$$= 4\tan^{-1} t - \frac{2t}{t^2+1} - 2\tan^{-1} t$$

$$= 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2}.$$

ゆえに

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\beta\to 1-0}\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2}\right]_{-1}^\beta$$

$$= \lim_{\beta\to 1-0}\left(2\tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{1-\beta^2}\right) = 2\cdot\frac{\pi}{2} = \pi.$$

(3)　$f(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$ は $(0,\pi]$ で連続なので，

$$\int_0^\pi \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)dx = \lim_{\alpha\to +0}\int_\alpha^\pi \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)dx$$

$$= \lim_{\alpha\to +0}\int_\alpha^\pi \left(\frac{\sin x}{x}\right)' dx = \lim_{\alpha\to +0}\left[\frac{\sin x}{x}\right]_\alpha^\pi$$

$$= -\lim_{\alpha\to +0}\frac{\sin\alpha}{\alpha} = -1.$$

(4)　$f(x) = \sin x\log(\sin x)$ は $\left(0, \dfrac{\pi}{2}\right]$ で連続なので，

$$\int_0^{\frac{\pi}{2}} \sin x\log(\sin x)\,dx = \lim_{\alpha\to +0}\int_\alpha^{\frac{\pi}{2}} \sin x\log(\sin x)\,dx.$$

ここで

$$\begin{aligned}\int \sin x \log(\sin x)\,dx &= -\cos x \log(\sin x) + \int \frac{\cos^2 x}{\sin x}\,dx \\ &= -\cos x \log(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx \\ &= -\cos x \log(\sin x) + \int \frac{dx}{\sin x} - \int \sin x\,dx \\ &= -\cos x \log(\sin x) + \int \frac{dx}{\sin x} + \cos x.\end{aligned}$$

ここで，$\tan\dfrac{x}{2} = t$ とおくと，$\sin x = \dfrac{2t}{t^2+1}$，$dx = \dfrac{2dt}{t^2+1}$．よって

$$\int \frac{dx}{\sin x} = \int \frac{t^2+1}{2t}\cdot\frac{2}{t^2+1}dt = \int \frac{dt}{t} = \log|t| = \log\left|\tan\frac{x}{2}\right|.$$

ゆえに

$$\begin{aligned}I(\alpha) &:= \int_\alpha^{\frac{\pi}{2}} \sin x \log(\sin x)\,dx = \left[-\cos x \log(\sin x) + \log\left|\tan\frac{x}{2}\right| + \cos x\right]_\alpha^{\frac{\pi}{2}} \\ &= \cos\alpha\log(\sin\alpha) - \log\left(\tan\frac{\alpha}{2}\right) - \cos\alpha \\ &= \cos\alpha\log\left(2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right) - \log\left(\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right) - \cos\alpha \\ &= \cos\alpha\log 2 + \cos\alpha\log\left(\cos\frac{\alpha}{2}\right) + (\cos\alpha - 1)\log\left(\sin\frac{\alpha}{2}\right) + \log\left(\cos\frac{\alpha}{2}\right) - \cos\alpha \\ &= \cos\alpha\log 2 + \cos\alpha\log\left(\cos\frac{\alpha}{2}\right) - \sin^2\frac{\alpha}{2}\log\left(\sin^2\frac{\alpha}{2}\right) + \log\left(\cos\frac{\alpha}{2}\right) - \cos\alpha.\end{aligned}$$

よって，$\lim_{x\to+0} x\log x = 0$ に注意すれば，

$$\int_0^{\frac{\pi}{2}} \sin x \log(\sin x)\,dx = \lim_{\alpha\to+0} I(\alpha) = \log 2 - 1.$$

(参考)　$\tan\dfrac{x}{2} = \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \dfrac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}} = \dfrac{\sin x}{2\cos^2\frac{x}{2}} = \dfrac{\sin x}{1+\cos x}$ を用いて，$I(\alpha)$ を変形すると，

$$\begin{aligned}I(\alpha) &= \cos\alpha\log(\sin\alpha) - \log\frac{\sin\alpha}{1+\cos\alpha} - \cos\alpha \\ &= (\cos\alpha - 1)\log\sin\alpha + \log(1+\cos\alpha) - \cos\alpha\end{aligned}$$

ここで，ロピタルの定理より，

$$\begin{aligned}\lim_{\alpha\to+0}(\cos\alpha - 1)\log(\sin\alpha) &= \lim_{\alpha\to+0}\frac{\log(\sin\alpha)}{\frac{1}{\cos\alpha - 1}} = \lim_{\alpha\to+0}\frac{\frac{\cos\alpha}{\sin\alpha}}{\frac{\sin\alpha}{(\cos\alpha-1)^2}} \\ &= \lim_{\alpha\to+0}\frac{\cos\alpha\,(\cos\alpha-1)^2}{\sin^2\alpha} = \lim_{\alpha\to+0}\frac{\cos\alpha\,(\cos\alpha-1)^2}{1-\cos^2\alpha} \\ &= \lim_{\alpha\to+0}\frac{\cos\alpha\,(1-\cos\alpha)}{1+\cos\alpha} = 0.\end{aligned}$$

ゆえに

$$\int_0^{\frac{\pi}{2}} \sin x \log(\sin x)\,dx = \lim_{\alpha\to+0} I(\alpha) = \log 2 - 1.$$

(5)　$f(x)=\dfrac{1}{\sqrt{x(3-x)}}$ は $(0,3)$ で連続なので，

$$\int_0^3 \frac{dx}{\sqrt{x(3-x)}}=\int_0^1 \frac{dx}{\sqrt{x(3-x)}}+\int_1^3 \frac{dx}{\sqrt{x(3-x)}}.$$

ここで，$\sqrt{\dfrac{x}{3-x}}=t\ \ (0<x<3)$ とおくと，

$$x=\frac{3t^2}{t^2+1},\quad dx=\frac{6t}{(t^2+1)^2}dt,\quad \frac{1}{\sqrt{x(3-x)}}=\frac{1}{x}\sqrt{\frac{x}{3-x}}=\frac{t(t^2+1)}{3t^2}.$$

よって

$$\int \frac{dx}{\sqrt{x(3-x)}}=\int \frac{t(t^2+1)}{3t^2}\cdot\frac{6t}{(t^2+1)^2}\,dt=2\int\frac{dt}{t^2+1}=2\tan^{-1}t=2\tan^{-1}\sqrt{\frac{x}{3-x}}.$$

ゆえに

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x(3-x)}}&=\lim_{\alpha\to+0}\int_\alpha^1 \frac{dx}{\sqrt{x(3-x)}}=\lim_{\alpha\to+0}\left[2\tan^{-1}\sqrt{\frac{x}{3-x}}\right]_\alpha^1\\&=\lim_{\alpha\to+0}2\left(\tan^{-1}\frac{1}{\sqrt{2}}-\tan^{-1}\sqrt{\frac{\alpha}{3-\alpha}}\right)=2\tan^{-1}\frac{1}{\sqrt{2}}.\end{aligned}$$

同様に

$$\begin{aligned}\int_1^\beta \frac{dx}{\sqrt{x(3-x)}}&=\lim_{\beta\to3-0}\int_1^\beta \frac{dx}{\sqrt{x(3-x)}}=\lim_{\beta\to3-0}\left[2\tan^{-1}\sqrt{\frac{x}{3-x}}\right]_1^\beta\\&=\lim_{\beta\to3-0}2\left(\tan^{-1}\sqrt{\frac{\beta}{3-\beta}}-\tan^{-1}\frac{1}{\sqrt{2}}\right)\\&=2\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{\sqrt{2}}\right)=\pi-2\tan^{-1}\frac{1}{\sqrt{2}}.\end{aligned}$$

以上より，

$$\int_0^3 \frac{dx}{\sqrt{x(3-x)}}=2\tan^{-1}\frac{1}{\sqrt{2}}+\pi-2\tan^{-1}\frac{1}{\sqrt{2}}=\pi.$$

(6)　$f(x)=\dfrac{\cos x}{\sqrt{\sin x}}$ は $\left(0,\dfrac{\pi}{2}\right]$ で連続なので，

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{\sin x}}\,dx&=\lim_{\alpha\to+0}\int_\alpha^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{\sin x}}\,dx=\lim_{\alpha\to+0}\left[2\sqrt{\sin x}\right]_\alpha^{\frac{\pi}{2}}\\&=\lim_{\alpha\to+0}2\left(1-\sqrt{\sin\alpha}\right)=2.\end{aligned}$$

(7)　部分積分を 2 回繰り返せば，

$$\begin{aligned}\int x^2e^{-x}\,dx&=-x^2e^{-x}+2\int xe^{-x}\,dx=-x^2e^{-x}-2xe^{-x}+2\int e^{-x}\,dx\\&=-(x^2+2x+2)e^{-x}.\end{aligned}$$

$f(x)=x^2e^{-x}$ は $[0,\infty)$ で連続なので，

$$\int_0^\infty x^2e^{-x}\,dx=\lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx=\lim_{\beta\to\infty}\Big[-(x^2+2x+2)e^{-x}\Big]_0^\beta$$

$$= \lim_{\beta\to\infty}\left\{2-(\beta^2+2\beta+2)e^{-\beta}\right\} = 2.$$

(8)　$f(x) = \dfrac{1}{1+x^2}$ は $(-\infty, \infty)$ で連続な偶関数なので，

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\int_0^{\infty}\frac{dx}{1+x^2} = 2\lim_{\beta\to\infty}\int_0^{\beta}\frac{dx}{1+x^2} = 2\lim_{\beta\to\infty}\Big[\tan^{-1}x\Big]_0^{\beta}$$
$$= 2\lim_{\beta\to\infty}\tan^{-1}\beta = 2\cdot\frac{\pi}{2} = \pi.$$

(9)　$f(x) = \dfrac{x^2}{(x^2+1)(x^2+4)}$ は $[0, \infty)$ で連続なので，

$$\int_0^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}\,dx = \lim_{\beta\to\infty}\int_0^{\beta}\frac{x^2}{(x^2+1)(x^2+4)}\,dx.$$

ここで

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

とおくと，

$$x^2 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B + D.$$

両辺の係数を比較すると，$A+C=0$，$B+D=1$，$4A+C=0$，$4B+D=0$．よって，$A=C=0$，$B=-\dfrac{1}{3}$，$D=\dfrac{4}{3}$．ゆえに

$$\int_0^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}\,dx = \lim_{\beta\to\infty}\frac{1}{3}\int_0^{\beta}\left(-\frac{1}{x^2+1}+\frac{4}{x^2+4}\right)dx$$
$$= \lim_{\beta\to\infty}\frac{1}{3}\left[-\tan^{-1}x + 2\tan^{-1}\frac{x}{2}\right]_0^{\beta}$$
$$= \lim_{\beta\to\infty}\frac{1}{3}\left(-\tan^{-1}\beta + 2\tan^{-1}\frac{\beta}{2}\right)$$
$$= \frac{1}{3}\left(-\frac{\pi}{2} + 2\cdot\frac{\pi}{2}\right) = \frac{\pi}{6}.$$

(10)　$e^x = t$ とおくと，$e^x\,dx = dt$．よって

$$\int\frac{dx}{e^x+4e^{-x}+5} = \int\frac{1}{t+\frac{4}{t}+5}\cdot\frac{dt}{t} = \int\frac{dt}{t^2+5t+4}$$
$$= \frac{1}{3}\int\left(\frac{1}{t+1}-\frac{1}{t+4}\right)dt = \frac{1}{3}\log\left|\frac{t+1}{t+4}\right|$$
$$= \frac{1}{3}\log\frac{e^x+1}{e^x+4}.$$

ここで，$f(x) = \dfrac{1}{e^x+4e^{-x}+5}$ は $[0, \infty)$ で連続なので，

$$\int_0^{\infty}\frac{dx}{e^x+4e^{-x}+5} = \lim_{\beta\to\infty}\int_0^{\beta}\frac{dx}{e^x+4e^{-x}+5} = \lim_{\beta\to\infty}\left[\frac{1}{3}\log\frac{e^x+1}{e^x+4}\right]_0^{\beta}$$
$$= \frac{1}{3}\lim_{\beta\to\infty}\left(\log\frac{e^{\beta}+1}{e^{\beta}+4} - \log\frac{2}{5}\right) = -\frac{1}{3}\log\frac{2}{5}.$$

(11) $x+\sqrt{x^2+1}=t$ とおくと，$x=\dfrac{t^2-1}{2t}$，$dx=\dfrac{t^2+1}{2t^2}\,dt$．よって

$$\begin{aligned}\int\frac{dx}{(x+\sqrt{x^2+1})^2}&=\int\frac{1}{t^2}\cdot\frac{t^2+1}{2t}\,dt=\frac{1}{2}\int\frac{t^2+1}{t^4}\,dt\\&=\frac{1}{2}\int\left(t^{-2}+t^{-4}\right)dt=\frac{1}{2}\left(-t^{-1}-\frac{1}{3}t^{-3}\right)\\&=-\frac{1}{2}\cdot\frac{1}{x+\sqrt{x^2+1}}-\frac{1}{3}\cdot\frac{1}{(x+\sqrt{x^2+1})^3}.\end{aligned}$$

ここで，$f(x)=\dfrac{1}{(x+\sqrt{x^2+1})^2}$ は $[0,\infty)$ で連続なので，

$$\begin{aligned}\int_0^\infty\frac{dx}{(x+\sqrt{x^2+1})^2}&=\lim_{\beta\to\infty}\int_0^\beta\frac{dx}{(x+\sqrt{x^2+1})^2}\\&=\lim_{\beta\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{x+\sqrt{x^2+1}}-\frac{1}{3}\cdot\frac{1}{(x+\sqrt{x^2+1})^3}\right]_0^\beta\\&=\lim_{\beta\to\infty}\left\{-\frac{1}{2}\cdot\frac{1}{\beta+\sqrt{\beta^2+1}}-\frac{1}{3}\cdot\frac{1}{(\beta+\sqrt{\beta^2+1})^3}+\frac{1}{2}+\frac{1}{6}\right\}\\&=\frac{2}{3}.\end{aligned}$$

(12) $f(x)=\left(x-\sqrt{x^2-1}\right)^4$ は $[1,\infty)$ で連続なので，

$$\int_1^\infty\left(x-\sqrt{x^2-1}\right)^4dx=\lim_{\beta\to\infty}\int_1^\beta\left(x-\sqrt{x^2-1}\right)^4dx.$$

ここで，$x-\sqrt{x^2-1}=t$ とおくと，$x=\dfrac{t^2+1}{2t}$，$dx=\dfrac{t^2-1}{2t^2}\,dt$．よって

$$\begin{aligned}\int\left(x-\sqrt{x^2-1}\right)^4dx&=\int t^4\cdot\frac{t^2-1}{2t^2}\,dt=\frac{1}{2}\int(t^4-t^2)\,dt=\frac{1}{2}\left(\frac{t^5}{5}-\frac{t^3}{3}\right)\\&=\frac{1}{10}\left(x-\sqrt{x^2-1}\right)^5-\frac{1}{6}\left(x-\sqrt{x^2-1}\right)^3.\end{aligned}$$

ゆえに

$$\begin{aligned}\int_1^\infty\left(x-\sqrt{x^2-1}\right)^4dx&=\lim_{\beta\to\infty}\left[\frac{1}{10}\left(x-\sqrt{x^2-1}\right)^5-\frac{1}{6}\left(x-\sqrt{x^2-1}\right)^3\right]_1^\beta\\&=\lim_{\beta\to\infty}\left\{\frac{1}{10}\left(\beta-\sqrt{\beta^2-1}\right)^5-\frac{1}{6}\left(\beta-\sqrt{\beta^2-1}\right)^3+\frac{1}{15}\right\}\\&=\lim_{\beta\to\infty}\left\{\frac{1}{10}\cdot\frac{1}{(\beta+\sqrt{\beta^2-1})^5}-\frac{1}{6}\cdot\frac{1}{(\beta+\sqrt{\beta^2-1})^3}\right\}+\frac{1}{15}\\&=\frac{1}{15}.\end{aligned}$$

12. (1) $f(x)=\dfrac{1}{\sqrt{(b-x)(x-a)}}$ は (a,b) で連続なので，

$$\int_a^b\frac{dx}{\sqrt{(b-x)(x-a)}}=\lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\int_\alpha^\beta\frac{dx}{\sqrt{(b-x)(x-a)}}\right)$$

である．ここで，$\sqrt{\dfrac{b-x}{x-a}}=t$ とおくと，

$$x=\frac{at^2+b}{t^2+1},\quad x-a=\frac{b-a}{t^2+1},\quad dx=-\frac{2(b-a)t}{(t^2+1)^2}\,dt$$

なので，

$$\begin{aligned}\int\frac{dx}{\sqrt{(b-x)(x-a)}}&=\int\frac{dx}{(x-a)\sqrt{\dfrac{b-x}{x-a}}}=-\int\frac{t^2+1}{(b-a)t}\cdot\frac{2(b-a)t}{(t^2+1)^2}\,dt\\&=-2\int\frac{dt}{t^2+1}=-2\tan^{-1}t=-2\tan^{-1}\sqrt{\frac{b-x}{x-a}}.\end{aligned}$$

を得る．よって，

$$\begin{aligned}\int_a^b\frac{dx}{\sqrt{(b-x)(x-a)}}&=\lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\left[-2\tan^{-1}\sqrt{\frac{b-x}{x-a}}\right]_\alpha^\beta\right)\\&=\lim_{\alpha\to a-0}2\tan^{-1}\sqrt{\frac{b-\alpha}{\alpha-a}}-\lim_{\beta\to b-0}2\tan^{-1}\sqrt{\frac{b-\beta}{\beta-a}}=\pi.\end{aligned}$$

別解． $f(x)=\dfrac{1}{\sqrt{(b-x)(x-a)}}$ は (a,b) で連続なので，

$$\begin{aligned}\int_a^b\frac{dx}{\sqrt{(b-x)(x-a)}}&=\int_a^b\frac{dx}{\sqrt{\left(\dfrac{b-a}{2}\right)^2-\left(x-\dfrac{a+b}{2}\right)^2}}\\&=\lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\int_\alpha^\beta\frac{dx}{\sqrt{\left(\dfrac{b-a}{2}\right)^2-\left(x-\dfrac{a+b}{2}\right)^2}}\right)\\&=\lim_{\alpha\to a-0}\left(\lim_{\beta\to b-0}\left[\sin^{-1}\frac{2x-(a+b)}{b-a}\right]_\alpha^\beta\right)\\&=\lim_{\beta\to b-0}\sin^{-1}\frac{2\beta-(a+b)}{b-a}-\lim_{\alpha\to a-0}\sin^{-1}\frac{2\alpha-(a+b)}{b-a}\\&=\frac{\pi}{2}+\frac{\pi}{2}=\pi.\end{aligned}$$

(2) $f(x)=\dfrac{1}{x^2+2x\cos\alpha+1}$ は $[0,\infty)$ で連続である．記述を簡単にするために，

$$I:=\int_0^\infty\frac{dx}{x^2+2x\cos\alpha+1}=\lim_{\beta\to\infty}\int_0^\beta\frac{dx}{x^2+2x\cos\alpha+1}$$

とおく．まず，$\alpha=0$ のときは，

$$I=\lim_{\beta\to\infty}\int_0^\beta\frac{dx}{(x+1)^2}=\lim_{\beta\to\infty}\left[-\frac{1}{x+1}\right]_0^\beta=\lim_{\beta\to\infty}\left(1-\frac{1}{\beta+1}\right)=1.$$

次に，$\alpha\neq0$ のときは，

$$I=\lim_{\beta\to\infty}\int_0^\beta\frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$$

$$= \lim_{\beta\to\infty} \left[\frac{1}{|\sin\alpha|} \tan^{-1} \frac{x+\cos\alpha}{|\sin\alpha|} \right]_0^{\beta}$$

$$= \lim_{\beta\to\infty} \frac{1}{|\sin\alpha|} \left(\tan^{-1} \frac{\beta+\cos\alpha}{|\sin\alpha|} - \tan^{-1} \frac{\cos\alpha}{|\sin\alpha|} \right)$$

$$= \frac{1}{|\sin\alpha|} \left(\frac{\pi}{2} - \tan^{-1} \frac{\cos\alpha}{|\sin\alpha|} \right)$$

$$= \begin{cases} \dfrac{1}{\sin\alpha} \left\{ \dfrac{\pi}{2} - \tan^{-1} \dfrac{\cos\alpha}{\sin\alpha} \right\} & (\sin\alpha > 0) \\ \dfrac{1}{\sin\alpha} \left\{ -\dfrac{\pi}{2} - \tan^{-1} \dfrac{\cos\alpha}{\sin\alpha} \right\} & (\sin\alpha < 0) \end{cases}$$

さて，

$$\theta_1 := \frac{\pi}{2} - \tan^{-1} \frac{\cos\alpha}{\sin\alpha}, \quad \theta_2 := -\frac{\pi}{2} - \tan^{-1} \frac{\cos\alpha}{\sin\alpha}$$

とおき，以下で場合を分けて，$\theta_1 = \theta_2 = \alpha$ となることを示す.

(i) $0 < \alpha < \dfrac{\pi}{2}$ のときは，$\sin\alpha > 0$，$\cos\alpha > 0$，$0 < \theta_1 < \dfrac{\pi}{2}$ である．さらに，

$$\frac{\cos\alpha}{\sin\alpha} = \tan\left(\frac{\pi}{2} - \theta_1\right) = \frac{\cos\theta_1}{\sin\theta_1}$$

より，$\tan\alpha = \tan\theta_1$ となる．よって，$\theta_1 = \alpha$.

(ii) $\alpha = \dfrac{\pi}{2}$ のときは，$\theta_1 = \dfrac{\pi}{2} - \tan^{-1} 0 = \dfrac{\pi}{2} = \alpha$.

(iii) $\dfrac{\pi}{2} < \alpha < \pi$ のときは，$\sin\alpha > 0$，$\cos\alpha <$，$\dfrac{\pi}{2} < \theta_1 < \pi$．さらに，(i) と同様にして，$\tan\alpha = \tan\theta_1$ となる．よって，$\theta_1 = \alpha$.

(iv) $-\dfrac{\pi}{2} < \alpha < 0$ のときは，$\sin\alpha < 0$，$\cos\alpha > 0$，$-\dfrac{\pi}{2} < \theta_2 < 0$．さらに，

$$\frac{\cos\alpha}{\sin\alpha} = \tan\left(-\frac{\pi}{2} - \theta_2\right) = \frac{-\sin\left(\dfrac{\pi}{2} + \theta_2\right)}{\cos\left(\dfrac{\pi}{2} + \theta_2\right)} = \frac{-\cos\theta_2}{-\sin\theta_2} = \frac{\cos\theta_2}{\sin\theta_2}$$

より，$\tan\alpha = \tan\theta_2$ となる．よって，$\theta_2 = \alpha$.

(v) $\alpha = -\dfrac{\pi}{2}$ のときは，$\theta_2 = -\dfrac{\pi}{2} - \tan^{-1} 0 = -\dfrac{\pi}{2} = \alpha$.

(vi) $-\pi < \alpha < -\dfrac{\pi}{2}$ のときは，$\sin\alpha < 0$，$\cos\alpha < 0$，$-\pi < \theta_2 < -\dfrac{\pi}{2}$．さらに，(iv) と同様にして，$\tan\alpha = \tan\theta_2$ となる．よって，$\theta_2 = \alpha$.

以上より，$\sin\alpha > 0$ のときは $\theta_1 = \alpha$ であり，$\sin\alpha < 0$ のときは $\theta_2 = \alpha$ となるので，$I = \dfrac{\alpha}{\sin\alpha}$ を得る.

13. (1)　$0 < x < \dfrac{\pi}{2}$ のとき，$0 < \sin x < 1$．よって，$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$．ゆえに，定理 3.6 の (4) より，(1) の不等式が成り立つ.

(2)　例 3.2.8 を用いて，(1) の不等式を書き直すと，

$$\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n+1)} < \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}\cdot\frac{\pi}{2} < \frac{2\cdot 4\cdot\cdots\cdot(2n-2)}{1\cdot 3\cdot\cdots\cdot(2n-1)}$$

となる．上式のすべての項に

$$\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}$$

を掛けると，(2) の不等式を得る．

(3)　(2) の不等式を変形すると，

$$\frac{2n}{2n+1} < \frac{\pi}{\dfrac{1}{n}\left\{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right\}^2} < 1$$

となる．よって，$n\to\infty$ とすると，$\dfrac{2n}{2n+1}\to 1$ より，

$$\pi = \lim_{n\to\infty}\frac{1}{n}\left\{\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right\}^2$$

を得る．ゆえに，

$$\sqrt{\pi} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{\{2\cdot 4\cdot\cdots\cdot(2n)\}^2}{(2n)!} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!}.$$

14.　以下では，求める面積を S で表す.

(1)　2 曲線 $y=\sin x$ と $y=\cos 2x$ の $0\leqq x\leqq\pi$ での交点の x 座標は $x=\dfrac{\pi}{6},\dfrac{5\pi}{6}$．よって，図形の対称性より，

$$\begin{aligned}S &= 2\left\{\int_0^{\frac{\pi}{6}}(\cos 2x-\sin x)\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\sin x-\cos 2x)\,dx\right\}\\ &= 2\left\{\left[\frac{\sin 2x}{2}+\cos x\right]_0^{\frac{\pi}{6}}+\left[-\cos x-\frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}\\ &= 2\left\{\left(\frac{3}{4}\sqrt{3}-1\right)+\frac{3}{4}\sqrt{3}\right\} = 3\sqrt{3}-2.\end{aligned}$$

(2)　2 曲線 $y=x^2$ と $\sqrt{x}+\sqrt{y}=2$ の $0\leqq x\leqq 4$ での交点の x 座標は $x=1$．よって

$$\begin{aligned}S &= \int_0^1\left\{\left(2-\sqrt{x}\right)^2-x^2\right\}dx = \int_0^1\left(4-4\sqrt{x}+x-x^2\right)dx\\ &= \left[4x-\frac{8}{3}x^{\frac{3}{2}}+\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 4-\frac{8}{3}+\frac{1}{2}-\frac{1}{3}=\frac{3}{2}.\end{aligned}$$

(3)　2 曲線 $y^2=4x$ と $x^2=4y$ の交点の x 座標は $x=0,4$．よって

$$S=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{x^3}{12}\right]_0^4=\frac{4}{3}4^{\frac{3}{2}}-\frac{1}{12}4^3=\frac{16}{3}.$$

(4)　媒介変数表示式が表す曲線と x 軸との交点の x 座標を求めるために，$y=2-t-t^2=0$ とおくと，$t=-2,1$．これを $x=2t+1$ に代入して，$x=-3,3$ を得る．$x=2t+1$，$y=2-t-t^2$ なので，$dx=2dt$ である．また，x が -3 から 3 まで動くとき，t は -2 から 1 まで動くので，

$$S=\int_{-3}^3 y\,dx=\int_{-2}^1\left(2-t-t^2\right)2\,dt=2\left[2t-\frac{t^2}{2}-\frac{t^3}{3}\right]_{-2}^1=9.$$

(5)　$x=t^2$，$y=t^3$ なので，$dx=2tdt$．また，x が 0 から 1 まで動くとき，t も 0 から 1 まで動く．よって，図形の対称性より

$$S=2\int_0^1 y\,dx=2\int_0^1 t^3\cdot 2t\,dt=4\int_0^1 t^4\,dt=4\left[\frac{t^5}{5}\right]_0^1=\frac{4}{5}.$$

(6)　曲線と x 軸との交点は $(\pm1,0)$，y 軸との交点は $(0,\pm1)$ である．$x=\cos^3 t$，$y=\sin^3 t$ なので，$dx = 3\cos^3 t\,(-\sin t)\,dt = -3\sin t\cos^2 t\,dt$．また，$x$ が 0 から 1 まで動くとき，t は $\dfrac{\pi}{2}$ から 0 まで動く．よって，図形の対称性より，

$$
\begin{aligned}
S &= 4\int_0^1 y\,dx = 4\int_{\frac{\pi}{2}}^0 \sin^3 t(-3\sin t\cos^3 t)\,dt = 12\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt \\
&= 12\left\{\int_0^{\frac{\pi}{2}} \sin^4 t\,dt - \int_0^{\frac{\pi}{2}} \sin^6 t\,dt\right\} \\
&= 12\left\{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\} \quad (\text{例 3.2.8 の公式}) \\
&= \frac{3\pi}{8}
\end{aligned}
$$

15.　以下では，求める面積を S で表す.

(1)　$2\sin^2\theta = 1-\cos 2\theta$ なので，

$$
S = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (2\sin\theta^2)\,d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1-\cos 2\theta)\,d\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{\pi}{2}+1.
$$

(2)　$\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1-\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} - 1$ なので，

$$
S = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta\,d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{\cos^2\theta} - 1\right)d\theta = \frac{1}{2}\Big[\tan\theta - \theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{\sqrt{3}} - \frac{\pi}{12}.
$$

(3)　$(1+\cos\theta)^2 = 1+2\cos\theta+\cos^2\theta = 1+2\cos\theta+\dfrac{1+\cos 2\theta}{2}$ なので，

$$
\begin{aligned}
S &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\
&= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta+\frac{\sin 2\theta}{2}}{2}\right]_0^{\frac{\pi}{2}} = \frac{3}{8}\pi + 1.
\end{aligned}
$$

16.　以下では，求める曲線の長さを L で表す.

(1)　$x=at^2$，$y=2at$ なので，$x'=2at$，$y'=2a$．よって，$(x')^2+(y')^2 = 4a^2t^2+4a^2 = 4a^2(1+t^2)$．ゆえに

$$
\begin{aligned}
L &= \int_0^1 \sqrt{4a^2(1+t^2)}\,dt = 2a\int_0^1 \sqrt{t^2+1}\,dt \\
&= 2a\left[\frac{1}{2}\left(t\sqrt{t^2+1} + \log\left|t+\sqrt{t^2+1}\right|\right)\right]_0^1 \quad (\text{例 3.1.6 (1) の公式}) \\
&= a\left\{\sqrt{2} + \log\left(1+\sqrt{2}\right)\right\}.
\end{aligned}
$$

(2)　$x = t\cos\dfrac{1}{t}$，$y = t\sin\dfrac{1}{t}$ なので，

$$
x' = \cos\frac{1}{t} + t\left(-\sin\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = \cos\frac{1}{t} + \frac{1}{t}\sin\frac{1}{t},
$$

$$y' = \sin\frac{1}{t} + t\left(\cos\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

よって

$$(x')^2 + (y')^2 = \left(\cos\frac{1}{t} + \frac{1}{t}\sin\frac{1}{t}\right)^2 + \left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right)^2 = \frac{t^2+1}{t^2}$$

より，

$$\sqrt{(x')^2 + (y')^2} = \frac{\sqrt{t^2+1}}{t} \quad (t > 0)$$

となる．ゆえに，

$$L = \int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt$$

である．$x = \sqrt{t^2+1}$ とおくと，$dx = \dfrac{t}{\sqrt{t^2+1}}\,dt$．また，$t$ が 1 から 2 まで動くとき，x は $\sqrt{2}$ から $\sqrt{5}$ まで動く．よって，

$$\frac{\sqrt{t^2+1}}{t}dt = \frac{t^2+1}{t^2}\cdot\frac{t}{\sqrt{t^2+1}}dt = \frac{x^2}{x^2-1}dx$$

に注意すると，

$$\begin{aligned} L &= \int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt = \int_{\sqrt{2}}^{\sqrt{5}} \frac{x^2}{x^2-1}\,dx = \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{x^2-1}\right)\,dx \\ &= \left[x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_{\sqrt{2}}^{\sqrt{5}} \\ &= \sqrt{5} - \sqrt{2} + \frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{\sqrt{5}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right\}. \end{aligned}$$

ここで，

$$\log\frac{\sqrt{5}-1}{\sqrt{5}+1} = \log\frac{\left(\sqrt{5}-1\right)^2}{5-1} = \log\frac{\left(\sqrt{5}-1\right)^2}{4} = 2\log\left(\sqrt{5}-1\right) - 2\log 2$$

$$\log\frac{\sqrt{2}-1}{\sqrt{2}+1} = \log\frac{\left(\sqrt{2}-1\right)^2}{2-1} = 2\log\left(\sqrt{2}-1\right)$$

なので，

$$L = \sqrt{5} - \sqrt{2} + \log\left(\sqrt{5}-1\right) - \log 2 - \log\left(\sqrt{2}-1\right) = \sqrt{5} - \sqrt{2} + \log\frac{\sqrt{5}-1}{2\left(\sqrt{2}-1\right)}.$$

(3)　$r = a\theta$ なので，$r' = a$．よって，$\sqrt{r^2 + (r')^2} = \sqrt{a^2\theta^2 + a^2} = a\sqrt{1+\theta^2}$．ゆえに

$$\begin{aligned} L &= a\int_0^{2\pi} \sqrt{\theta^2+1}\,d\theta \\ &= a\left[\frac{1}{2}\left(\theta\sqrt{\theta^2+1} + \log\left|\theta + \sqrt{\theta^2+1}\right|\right)\right]_0^{2\pi} \quad (\text{例 3.1.6 (1) の公式}) \\ &= a\left\{\pi\sqrt{4\pi^2+1} + \frac{1}{2}\log\left(2\pi + \sqrt{4\pi^2+1}\right)\right\} \end{aligned}$$

(4)　$r = a\sin\theta$ なので，$r' = a\cos\theta$．よって，$\sqrt{r^2 + (r')^2} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$．ゆえに

$$L = \int_0^{\pi} a\,d\theta = \pi a.$$

17. 以下では，求める立体の体積を V で表す．

(1) xy-平面と平行な平面 $z=z$ での切り口は楕円 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1-\dfrac{z^4}{c^4}$ なので，その面積 $S(z)$ は

$$S(z)=\pi ab\left(1-\frac{z^4}{c^4}\right)$$

である．よって，図形の対称性より，

$$V=2\int_0^c S(z)\,dz=2\pi ab\int_0^c\left(1-\frac{z^4}{c^4}\right)dz=2\pi ab\left[z-\frac{z^5}{5c^4}\right]_0^c=\frac{8}{5}\pi abc.$$

(2) yz-平面と平行な平面 $x=x$ での切り口は，1 辺が $2\sqrt{a^2-x^2}$ の正方形なので，その面積 $S(x)$ は，$S(x)=4\left(a^2-x^2\right)$．よって，図形の対称性より，

$$V=2\int_0^a S(x)\,dx=8\int_0^a\left(a^2-x^2\right)dx=8\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{16}{3}a^3.$$

18. 以下では，求める回転体の体積を V で表す．

(1) 曲線 $y=\sqrt{x}$ と直線 $y=\dfrac{x}{2}$ の交点の x 座標は $x=0,4$ である．よって

$$\begin{aligned}V&=\pi\int_0^4\left(\sqrt{x}\right)^2dx-\pi\int_0^4\left(\frac{x}{2}\right)^2dx\\&=\pi\int_0^4 x\,dx-\frac{\pi}{4}\int_0^4 x^2dx=\pi\left[\frac{x^2}{2}\right]_0^4-\frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^4=\frac{8}{3}\pi.\end{aligned}$$

(2) $\tan^2 x=\dfrac{\sin^2 x}{\cos^2 x}=\dfrac{1-\cos^2 x}{\cos^2 x}=\dfrac{1}{\cos^2 x}-1$ なので，

$$V=\pi\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=\pi\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)dx=\pi\Big[\tan x-x\Big]_0^{\frac{\pi}{4}}=\pi-\frac{\pi^2}{4}.$$

(3) $V=\pi\displaystyle\int_{\sqrt{2}}^3\left(x^2-2\right)dx=\pi\left[\frac{x^3}{3}-2x\right]_{\sqrt{2}}^3=\pi\left(3+\frac{4\sqrt{2}}{3}\right).$

(4) $x=t^2$，$y=t^3$ なので，$dx=2tdt$，$y^2=t^6$．また，x が 0 から 1 まで動くとき，t も 0 から 1 まで動く．よって

$$V=\pi\int_0^1 y^2dx=\pi\int_0^1 t^6\cdot 2t\,dt=2\pi\int_0^1 t^7dt=2\pi\left[\frac{t^8}{8}\right]_0^1=\frac{\pi}{4}.$$

(5) $x=t-1$，$y=4t-t^2$ なので，$dx=dt$，$y^2=\left(4t-t^2\right)^2=16t^2-8t^3+t^4$．また，$x$ が 0 から 2 まで動くとき，t は 1 から 3 まで動く．よって，

$$V=\pi\int_0^2 y^2dx=\pi\int_1^3\left(16t^2-8t^3+t^4\right)dt=\pi\left[\frac{16}{3}t^3-2t^4+\frac{t^5}{5}\right]_1^3=\frac{406}{15}\pi.$$