3　積分

3.3　広義積分

問 1.　(1)　$\dfrac{1}{\sqrt[3]{x}}$ は $(0,2]$ で連続．よって

$$\int_0^2 \frac{dx}{\sqrt[3]{x}} = \lim_{\alpha\to+0}\int_\alpha^2 \frac{dx}{\sqrt[3]{x}} = \lim_{\alpha\to+0}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_\alpha^2 = \lim_{\alpha\to+0}\frac{3}{2}\left(2^{\frac{2}{3}} - \alpha^{\frac{2}{3}}\right) = \frac{3}{2}\sqrt[3]{4}.$$

(2)　$\dfrac{1}{\sqrt{2-x}}$ は $[0,2)$ で連続．よって

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{2-x}} &= \lim_{\beta\to 2-0}\int_0^\beta \frac{dx}{\sqrt{2-x}} = \lim_{\beta\to 2-0}\left[-2\sqrt{2-x}\right]_0^\beta \\ &= \lim_{\beta\to 2-0} 2\left(\sqrt{2}-\sqrt{2-\beta}\right) = 2\sqrt{2}.\end{aligned}$$

(3)　$\dfrac{\log x}{x}$ は $(0,1]$ で連続．よって

$$\begin{aligned}\int_0^1 \frac{\log x}{x}\,dx &= \lim_{\alpha\to+0}\int_\alpha^1 \frac{\log x}{x}\,dx = \lim_{\alpha\to+0}\left[\frac{1}{2}(\log x)^2\right]_\alpha^1 \\ &= -\lim_{\alpha\to+0}\frac{1}{2}(\log\alpha)^2 = -\infty.\end{aligned}$$

(4)　$\dfrac{1}{\sqrt{(x-1)(2-x)}}$ は $(1,2)$ で連続．よって

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta \frac{dx}{\sqrt{(x-1)(2-x)}}\,dx\right) \\ &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\int_\alpha^\beta \frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx\right) \\ &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to2-0}\left[\sin^{-1}(2x-3)\right]_\alpha^\beta\right) \\ &= \lim_{\beta\to2-0}\sin^{-1}(2\beta-3) - \lim_{\alpha\to1+0}\sin^{-1}(2\alpha-3) \\ &= \sin^{-1}1 - \sin^{-1}(-1) = \pi.\end{aligned}$$

(5)　$\dfrac{1}{\sin x\cos x}$ は $\left(0,\dfrac{\pi}{2}\right)$ で連続．よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x\cos x} = \int_0^{\frac{\pi}{2}} \frac{2}{\sin 2x}\,dx = \lim_{\alpha\to+0}\left(\lim_{\beta\to\frac{\pi}{2}-0}\int_\alpha^\beta \frac{2}{\sin 2x}\,dx\right).$$

ここで

$$\begin{aligned}\int \frac{dx}{\sin 2x} &= \int \frac{\sin 2x}{\sin^2 2x}\,dx = \int \frac{\sin 2x}{1-\cos^2 2x}\,dx \\ &= \int\left(\frac{\sin 2x}{1-\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right)dx \\ &= \frac{1}{2}\log\frac{1-\cos 2x}{1+\cos 2x}.\end{aligned}$$

よって

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x \cos x} = \lim_{\beta \to \frac{\pi}{2}-0} \log \frac{1-\cos 2\beta}{1+\cos 2\beta} - \lim_{\alpha \to +0} \log \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$
$$= \lim_{\beta \to \frac{\pi}{2}-0} \log \frac{1-\cos 2\beta}{1+\cos 2\beta} + \lim_{\alpha \to +0} \log \frac{1+\cos 2\alpha}{1-\cos 2\alpha} = \infty.$$

(6) $\dfrac{1}{\sqrt{|x(x-2)|}}$ は $(0,2)$ と $(2,3]$ で連続．よって

$$\int_0^3 \frac{dx}{\sqrt{|x(x-2)|}} = \int_0^2 \frac{dx}{\sqrt{|x(x-2)|}} + \int_2^3 \frac{dx}{\sqrt{|x(x-2)|}}$$
$$= \int_0^2 \frac{dx}{\sqrt{-x(x-2)}} + \int_2^3 \frac{dx}{\sqrt{x(x-2)}}.$$

ここで

$$\int_0^2 \frac{dx}{\sqrt{-x(x-2)}} = \lim_{\alpha \to +0} \left(\lim_{\beta \to 2-0} \int_\alpha^\beta \frac{dx}{\sqrt{2x-x^2}} \right)$$
$$= \lim_{\alpha \to +0} \left(\lim_{\beta \to 2-0} \int_\alpha^\beta \frac{dx}{\sqrt{1-(x-1)^2}} \right)$$
$$= \lim_{\alpha \to +0} \left(\lim_{\beta \to 2-0} \Big[\sin^{-1}(x-1) \Big]_\alpha^\beta \right)$$
$$= \lim_{\beta \to 2-0} \sin^{-1}(\beta-1) - \lim_{\alpha \to +0} \sin^{-1}(\alpha-1)$$
$$= \sin^{-1} 1 - \sin^{-1}(-1) = \pi.$$

$$\int_2^3 \frac{dx}{\sqrt{x(x-2)}} = \lim_{\alpha \to 2+0} \int_\alpha^3 \frac{dx}{\sqrt{x^2-2x}} = \lim_{\alpha \to 2+0} \int_\alpha^3 \frac{dx}{\sqrt{(x-1)^2-1}}$$
$$= \lim_{\alpha \to 2+0} \left[\log \left| x-1+\sqrt{x^2-2x} \right| \right]_\alpha^3$$
$$= \lim_{\alpha \to 2+0} \left\{ \log\left(2+\sqrt{3}\right) - \log\left(\alpha-1+\sqrt{\alpha^2-2\alpha}\right) \right\}$$
$$= \log\left(2+\sqrt{3}\right).$$

以上より

$$\int_0^3 \frac{dx}{\sqrt{|x(x-2)|}} = \pi + \log\left(2+\sqrt{3}\right).$$

問 2. $\dfrac{1}{x^p}$ は $[1,\infty)$ で連続．ゆえに，$p>1$ のとき，

$$\int_1^\infty \frac{dx}{x^p} = \lim_{\beta \to \infty} \int_1^\beta \frac{dx}{x^p} = \lim_{\beta \to \infty} \left[\frac{1}{1-p} x^{1-p} \right]_1^\beta = \lim_{\beta \to \infty} \frac{1}{1-p} \left(\frac{1}{\beta^{p-1}} - 1 \right) = \frac{1}{p-1}.$$

次に，$p=1$ のとき，

$$\int_1^\infty \frac{dx}{x} = \lim_{\beta \to \infty} \int_1^\beta \frac{dx}{x} = \lim_{\beta \to \infty} \Big[\log x \Big]_1^\beta = \lim_{\beta \to \infty} \log \beta = \infty.$$

また，$p<1$ のとき，

$$\int_1^\infty \frac{dx}{x^p} = \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x^p} = \lim_{\beta\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^\beta = \lim_{\beta\to\infty}\frac{1}{1-p}\left(\beta^{1-p}-1\right) = \infty.$$

問 3. (1) $\dfrac{1}{x(1+x^2)}$ は $[1,\infty)$ で連続．よって

$$\begin{aligned}\int_1^\infty \frac{dx}{x(1+x^2)} &= \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x(1+x^2)} = \lim_{\beta\to\infty}\int_1^\beta\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx \\ &= \lim_{\beta\to\infty}\left[\log x - \frac{1}{2}\log\left(1+x^2\right)\right]_1^\beta \\ &= \frac{1}{2}\log 2 + \lim_{\beta\to\infty}\left\{\log\beta - \frac{1}{2}\log\left(1+\beta^2\right)\right\} = \frac{1}{2}\log 2.\end{aligned}$$

(2) $\dfrac{x^3}{1+x^4}$ は $[0,\infty)$ で連続．よって

$$\begin{aligned}\int_0^\infty \frac{x^3}{1+x^4}\,dx &= \lim_{\beta\to\infty}\int_0^\beta \frac{x^3}{1+x^4}\,dx = \lim_{\beta\to\infty}\left[\frac{1}{4}\log\left(1+x^4\right)\right]_0^\beta \\ &= \lim_{\beta\to\infty}\frac{1}{4}\log\left(1+\beta^4\right) = \infty.\end{aligned}$$

(3) xe^{-x^2} は $[0,\infty)$ で連続．よって

$$\int_0^\infty xe^{-x^2}dx = \lim_{\beta\to\infty}\int_0^\beta xe^{-x^2}dx = \lim_{\beta\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^\beta = \lim_{\beta\to\infty}\frac{1}{2}\left(1-e^{-\beta^2}\right) = \frac{1}{2}.$$

(4) $\dfrac{1}{x\sqrt{x^2-1}}$ は $(1,\infty)$ で連続．よって

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{\alpha\to1+0}\left(\lim_{\beta\to\infty}\int_\alpha^\beta \frac{dx}{x\sqrt{x^2-1}}\right).$$

ここで，$\sqrt{x^2-1}=t$ とおくと，$t^2=x^2-1$，$t\,dt = x\,dx$．よって

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{x}{x^2\sqrt{x^2-1}}\,dx = \int \frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{x^2-1}.$$

ゆえに

$$\begin{aligned}\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} &= \lim_{\alpha\to1+0}\left(\lim_{\beta\to\infty}\left[\tan^{-1}\sqrt{x^2-1}\right]_\alpha^\beta\right) \\ &= \lim_{\beta\to\infty}\tan^{-1}\sqrt{\beta^2-1} - \lim_{\alpha\to1+0}\tan^{-1}\sqrt{\alpha^2-1} \\ &= \frac{\pi}{2}.\end{aligned}$$

(5) ここで，$\dfrac{\log(1+x^2)}{x^2}$ は $(0,\infty)$ で連続．よって

$$\int_0^\infty \frac{\log\left(1+x^2\right)}{x^2}\,dx = \lim_{\alpha\to+0}\left(\lim_{\beta\to\infty}\int_\alpha^\beta \frac{\log\left(1+x^2\right)}{x^2}\,dx\right).$$

ここで

$$\int \frac{\log\left(1+x^2\right)}{x^2}\,dx = \int\left(-\frac{1}{x}\right)'\log\left(1+x^2\right)dx$$

$$= -\frac{1}{x}\log(1+x^2) + \int \frac{1}{x}\cdot\frac{2x}{1+x^2}\,dx$$
$$= -\frac{1}{x}\log(1+x^2) + 2\int \frac{dx}{1+x^2}$$
$$= -\frac{1}{x}\log(1+x^2) + 2\tan^{-1}x.$$

よって，ロピタルの定理を用いて極限を計算すると，

$$\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx = \lim_{\alpha\to+0}\left(\lim_{\beta\to\infty}\int_\alpha^\beta \frac{\log(1+x^2)}{x^2}\,dx\right)$$
$$= \lim_{\alpha\to+0}\left(\lim_{\beta\to\infty}\left[-\frac{1}{x}\log(1+x^2)+2\tan^{-1}x\right]_\alpha^\beta\right)$$
$$= \lim_{\beta\to\infty}\left\{-\frac{1}{\beta}\log(1+\beta^2)+2\tan^{-1}\beta\right\}$$
$$- \lim_{\alpha\to+0}\left\{-\frac{1}{\alpha}\log(1+\alpha^2)+2\tan^{-1}\alpha\right\}$$
$$= -\lim_{\beta\to\infty}\frac{\log(1+\beta^2)}{\beta} + \pi + \lim_{\alpha\to+0}\frac{\log(1+\alpha^2)}{\alpha}$$
$$= -\lim_{\beta\to\infty}\frac{2\beta}{\beta^2+1} + \pi + \lim_{\alpha\to+0}\frac{2\alpha}{\alpha^2+1} = \pi.$$

(6)　$e^{-ax}\sin bx$ は $[0,\infty)$ で連続．よって

$$\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-ax}\sin bx\,dx.$$

ここで

$$\int e^{-ax}\sin bx\,dx = \int\left(-\frac{1}{a}e^{-ax}\right)'\sin bx\,dx$$
$$= -\frac{1}{a}e^{-ax}\sin bx + \frac{b}{a}\int e^{-ax}\cos bx\,dx$$
$$= -\frac{1}{a}e^{-ax}\sin bx - \frac{b}{a^2}e^{-ax}\cos bx - \frac{b^2}{a^2}\int e^{-ax}\sin bx\,dx.$$

よって

$$\int e^{-ax}\sin bx\,dx = -\frac{e^{-ax}}{a^2+b^2}(a\sin bx + b\cos bx).$$

ゆえに

$$\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{\beta\to\infty}\left[-\frac{e^{-ax}}{a^2+b^2}(a\sin bx+b\cos bx)\right]_0^\beta$$
$$= -\lim_{\beta\to\infty}\left\{-\frac{e^{-a\beta}}{a^2+b^2}(a\sin b\beta + b\cos b\beta)\right\} + \frac{b}{a^2+b^2}$$
$$= \frac{b}{a^2+b^2}.$$

問 4.　(1)　e^{-x} は $[0,\infty)$ で連続．よって

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-x}\,dx = \lim_{\beta\to\infty}\left[-e^{-x}\right]_0^\beta = \lim_{\beta\to\infty}\left(1-e^{-\beta}\right) = 1.$$

(2)　x^2e^{-x} は $[0,\infty)$ で連続．よって

$$\Gamma(3)=\int_0^\infty x^2e^{-x}\,dx=\lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx.$$

ここで

$$\begin{aligned}\int x^2e^{-x}\,dx&=-x^2e^{-x}+2\int xe^{-x}\,dx=-x^2e^{-x}-2xe^{-x}+2\int e^{-x}\,dx\\&=-(x^2+2x+2)e^{-x}.\end{aligned}$$

よって

$$\begin{aligned}\Gamma(3)&=\int_0^\infty x^2e^{-x}\,dx=\lim_{\beta\to\infty}\int_0^\beta x^2e^{-x}\,dx=\lim_{\beta\to\infty}\Big[-(x^2+2x+2)e^{-x}\Big]_0^\beta\\&=\lim_{\beta\to\infty}\Big\{2-(\beta^2+2\beta+2)e^{-\beta}\Big\}=2.\end{aligned}$$

(3)　$\dfrac{1}{\sqrt{x(1-x)}}$ は $(0,1)$ で連続．よって

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1\frac{dx}{\sqrt{x(1-x)}}=\lim_{\alpha\to+0}\left(\lim_{\beta\to1-0}\int_\alpha^\beta\frac{dx}{\sqrt{x(1-x)}}\right).$$

ここで

$$\int\frac{dx}{\sqrt{x(1-x)}}=\int\frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}=\sin^{-1}(2x-1).$$

よって

$$\begin{aligned}B\left(\frac{1}{2},\frac{1}{2}\right)&=\lim_{\alpha\to+0}\left(\lim_{\beta\to1-0}\Big[\sin^{-1}(2x-1)\Big]_\alpha^\beta\right)\\&=\lim_{\beta\to1-0}\sin^{-1}(2\beta-1)-\lim_{\alpha\to+0}\sin^{-1}(2\alpha-1)\\&=\sin^{-1}1-\sin^{-1}(-1)=\pi.\end{aligned}$$

(4)　$\dfrac{(1-x)^2}{\sqrt{x}}$ は $(0,1]$ で連続．よって

$$B\left(\frac{1}{2},3\right)=\int_0^1\frac{(1-x)^2}{\sqrt{x}}\,dx=\lim_{\alpha\to+0}\int_\alpha^1\frac{(1-x)^2}{\sqrt{x}}\,dx.$$

ここで

$$\begin{aligned}\int\frac{(1-x)^2}{\sqrt{x}}\,dx&=\int\frac{x^2-2x+1}{\sqrt{x}}\,dx=\int\left(x^{\frac{3}{2}}-2x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx\\&=\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}.\end{aligned}$$

よって

$$B\left(\frac{1}{2},3\right)=\int_0^1\frac{(1-x)^2}{\sqrt{x}}\,dx=\lim_{\alpha\to+0}\int_\alpha^1\frac{(1-x)^2}{\sqrt{x}}\,dx$$

$$= \lim_{\alpha\to+0}\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_\alpha^1$$

$$= \frac{2}{5} - \frac{4}{3} + 2 - \lim_{\alpha\to+0}\left(\frac{2}{5}\alpha^{\frac{5}{2}} - \frac{4}{3}\alpha^{\frac{3}{2}} + 2\alpha^{\frac{1}{2}}\right) = \frac{16}{15}.$$

問 5. (1) $e^{-x}x^s$ は $[0,\infty)$ で連続．よって

$$\Gamma(s+1) = \int_0^\infty e^{-x}x^s\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-x}x^s\,dx.$$

ここで

$$\int e^{-x}x^s\,dx = -e^{-x}x^s + s\int e^{-x}x^{s-1}\,dx.$$

ロピタルの定理を繰り返し用いると

$$\lim_{x\to\infty} e^{-x}x^s = \lim_{x\to\infty}\frac{x^s}{e^x} = \lim_{x\to\infty}\frac{sx^{s-1}}{e^x} = 0$$

なので,

$$\Gamma(s+1) = \lim_{\beta\to\infty}\left(\left[-e^{-x}x^s\right]_0^\beta + s\int_0^\beta e^{-x}x^{s-1}\,dx\right) = s\Gamma(s).$$

(2) $p>0$，$q>1$ のとき，$x^{p-1}(1-x)^{q-1}$ は $(0,1]$ で連続．よって

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \lim_{\alpha\to+0}\int_\alpha^1 x^{p-1}(1-x)^{q-1}\,dx.$$

ここで

$$\int x^{p-1}(1-x)^{q-1}\,dx = \frac{1}{p}x^p(1-x)^{q-1} + \frac{q-1}{p}\int x^p(1-x)^{q-2}\,dx.$$

よって

$$B(p,q) = \lim_{\alpha\to+0}\left(\left[\frac{1}{p}x^p(1-x)^{q-1}\right]_\alpha^1 + \frac{q-1}{p}\int_\alpha^1 x^p(1-x)^{q-2}\,dx\right)$$

$$= -\lim_{\alpha\to+0}\frac{1}{p}\alpha^p(1-\alpha)^{q-1} + \frac{q-1}{p}\int_0^1 x^p(1-x)^{q-2}\,dx$$

$$= \frac{q-1}{p}B(p+1,q-1).$$

(3) (2) を繰り返し用いると

$$B(m,n) = \frac{n-1}{m}B(m+1,n-1) = \frac{(n-1)(n-2)}{m(m+1)}B(m+2,n-2)$$

$$= \frac{(n-1)(n-2)\cdots 1}{m(m+1)\cdots(m+n-2)}B(m+n-1,1).$$

ここで,

$$B(m+n-1,1) = \int_0^1 x^{m+n-2}\,dx = \left[\frac{1}{m+n-1}x^{m+n-1}\right]_0^1 = \frac{1}{m+n-1}.$$

よって

$$B(m,n) = \frac{(n-1)!}{m(m+1)\cdots(m+n-1)} = \frac{(m-1)!(n-1)!}{(m-1)!m(m+1)\cdots(m+n-1)}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}.$$

(4)　$x = \sin^2\theta$ とおくと，$dx = 2\sin\theta\cos\theta\,dt$．よって

$$\begin{aligned} B(p,q) &= \int_0^{\frac{\pi}{2}} \sin^{2p-2}\theta(1-\sin^2\theta)^{q-1} 2\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta|\cos\theta|^{2q-2}\cos\theta\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta. \end{aligned}$$