3 積分

3.1 不定積分

問 1. (1) $\displaystyle\int \frac{dx}{\cos^2(3x+1)} = \frac{1}{3}\tan(3x+1)$

(2) $\displaystyle\int \frac{dx}{\sqrt{4-(2x+1)^2}} = \frac{1}{2}\sin^{-1}\frac{2x+1}{2} = \frac{1}{2}\sin^{-1}\left(x+\frac{1}{2}\right)$

(3) $\displaystyle\int \frac{dx}{x^2+4x+6} = \int \frac{dx}{(x+2)^2+(\sqrt{2}\,)^2} = \frac{1}{\sqrt{2}}\,\tan^{-1}\frac{x+2}{\sqrt{2}}$

(4) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(\cos x)'}{\cos x}\,dx = -\log|\cos x|$

(5) $\displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx = \int \frac{(x+\sin x)'}{x+\sin x}\,dx = \log|x+\sin x|$

(6) $\displaystyle\int x^2(x^3+1)^{\frac{5}{2}}dx = \frac{1}{3}\int (x^3+1)^{\frac{5}{2}}(x^3+1)'dx = \frac{1}{3}\cdot\frac{1}{\frac{5}{2}+1}\,(x^3+1)^{\frac{5}{2}+1} = \frac{2}{21}\,(x^3+1)^{\frac{7}{2}}$

(7) $$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx &= \frac{1}{2}\int (x^2+2x+3)^{-\frac{1}{2}}(x^2+2x+3)'\,dx \\ &= \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}(x^2+2x+3)^{-\frac{1}{2}+1} = \sqrt{x^2+2x+3}\end{aligned}$$

(8) $\displaystyle\int \frac{(\log x)^7}{x}\,dx = \int (\log x)^7(\log x)'\,dx = \frac{1}{8}\,(\log x)^8$

(9) $$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= \int (1-x^2)^{-\frac{1}{2}}\left\{-\frac{1}{2}(1-x^2)\right\}'dx = -\frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}\,(1-x^2)^{-\frac{1}{2}+1} \\ &= -\sqrt{1-x^2}\end{aligned}$$

問 2. (1) $\displaystyle\int (x-3)(2x-1)\,dx = \int (2x^2-7x+3)\,dx = \frac{2}{3}x^3-\frac{7}{2}x^2+3x.$

(2) $\displaystyle\int \left(x+\frac{1}{x}\right)^2 dx = \int \left(x^2+2+\frac{1}{x^2}\right)dx = \frac{1}{3}x^3+2x-\frac{1}{x}.$

(3) $$\begin{aligned}\int \frac{(x-1)(\sqrt{x}-2)}{x^2}\,dx &= \int \frac{x^{\frac{3}{2}}-2x-x^{\frac{1}{2}}+2}{x^2}\,dx = \int \left(x^{-\frac{1}{2}}-\frac{2}{x}-x^{-\frac{3}{2}}+\frac{2}{x^2}\right)dx \\ &= 2\sqrt{x}-2\log|x|+2x^{-\frac{1}{2}}-\frac{2}{x} = 2\left(\sqrt{x}+\frac{1}{\sqrt{x}}-\frac{1}{x}-\log x\right)\end{aligned}$$

(4) $\displaystyle\int \left(e^x+e^{-x}\right)^2 dx = \int \left(e^{2x}+2+e^{-2x}\right)dx = \frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x} = \sinh 2x+2x$

(5) $\displaystyle\int \left(2^{x+1}+3^{x+1}\right)dx = \frac{2^{x+1}}{\log 2}+\frac{3^{x+1}}{\log 3}$

(6) $\displaystyle\int \frac{dx}{\cos^2 x\sin^2 x} = \int \frac{4}{\sin^2 2x}\,dx = -4\cdot\frac{1}{2}\cot 2x = -2\cot 2x$

別解. $\displaystyle\int \frac{dx}{\cos^2 x\sin^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\cos^2 x\sin^2 x}\,dx = \int \left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)dx = \tan x-\cot x$

(7) $\displaystyle\int \cos^2 \frac{x}{2}\,dx = \int \frac{1}{2}(1+\cos x)\,dx = \frac{1}{2}(x+\sin x)$

(8) $$\int \sin^3 x\,dx = \int \sin x \cdot \sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx = \int \sin x\,dx - \int \sin x\cos^2 x\,dx$$
$$= -\cos x + \frac{1}{3}\cos^3 x$$

別解. $\sin 3x = 3\sin x - 4\sin^3 x$ より，

$$\int \sin^3 x\,dx = \frac{1}{4}\int (3\sin x - \sin 3x)\,dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x.$$

(9) $$\int \sin x\cos 2x\,dx = \int \sin x(2\cos^2 x - 1)\,dx = 2\int \sin x\cos^2 x\,dx - \int \sin x\,dx$$
$$= -\frac{2}{3}\cos^3 x + \cos x$$

別解. $2\sin x\cos 2x = \sin(x+2x) + \sin(x-2x)$ より，

$$\int \sin x\cos 2x\,dx = \frac{1}{2}\int (\sin 3x - \sin x)\,dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x.$$

(10) $$\int \frac{dx}{\tan^2 x} = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - 1\right)dx = -\frac{1}{\tan x} - x$$

(11) $$\int \frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\,dx = \int \frac{(x+\sqrt{x^2-1})^2}{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}\,dx = \int (x+\sqrt{x^2-1}\,)^2 dx$$
$$= \int \left(2x^2 - 1 + 2x\sqrt{x^2-1}\right)\,dx = \frac{2}{3}\,x^3 - x + \frac{2}{3}(x^2-1)^{\frac{3}{2}}$$

(12) $$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx = \int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}\cdot\sqrt{1-x^2}}\,dx = \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\right)dx$$
$$= \sin^{-1} x + \log\left(x + \sqrt{x^2+1}\right)$$

問 3. (1) $2x+1=t$ とおくと，$dx = \frac{1}{2}\,dt$．よって

$$\int \frac{x^2}{(2x+1)^2}\,dx = \int \frac{1}{t^2}\cdot\frac{1}{4}(t-1)^2\cdot\frac{1}{2}\,dt = \frac{1}{8}\int \frac{t^2-2t+1}{t^2}\,dt$$
$$= \frac{1}{8}\int \left(1 - \frac{2}{t} + \frac{1}{t^2}\right)dt = \frac{1}{8}\left(t - 2\log|t| - \frac{1}{t}\right)$$
$$= \frac{1}{8}\left(2x+1 - 2\log|2x+1| - \frac{1}{2x+1}\right)$$
$$= \frac{1}{4}\left\{\frac{2x(x+1)}{2x+1} - \log|2x+1|\right\}.$$

参考. $x^2 = \dfrac{1}{4}\Big\{(2x+1)^2 - 2(2x+1) + 1\Big\}$ より，

$$\int \frac{x^2}{(2x+1)^2}\,dx = \frac{1}{4}\int \left\{1 - \frac{2}{2x+1} + \frac{1}{(2x+1)^2}\right\}dx$$

$$= \frac{1}{4}\left\{x - \log|2x+1| - \frac{1}{2(2x+1)}\right\}$$

$$= \frac{1}{8}\left(\frac{4x^2+2x-1}{2x+1} - 2\log|2x+1|\right).$$

(2)　$x^2 = t$ とおくと，$x\,dx = \frac{1}{2}\,dt$．よって

$$\int \frac{x}{\sqrt{x^4-2}}\,dx = \int \frac{1}{\sqrt{t^2-2}} \cdot \frac{1}{2}\,dt = \frac{1}{2}\ \log\left|t + \sqrt{t^2-2}\right| = \frac{1}{2}\ \log\left(x^2 + \sqrt{x^4-2}\right).$$

(3)　$x + \frac{1}{2} = t$ とおくと，$dx = dt$．よって

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

(4)　$\sqrt{1+x^2} = t$ とおくと，$x^2 = t^2 - 1$，$x\,dx = t\,dt$．よって

$$\int x^3\sqrt{1+x^2}\,dx = \int (t^2-1)t \cdot t\,dt = \int (t^4 - t^2)\,dt = \frac{1}{5}t^5 - \frac{1}{3}t^3$$

$$= \frac{1}{15}(3t^2-5)t^3 = \frac{1}{15}(3x^2-2)(x^2+1)\sqrt{x^2+1}.$$

参考.
$$\int x^3\sqrt{1+x^2}\,dx = \frac{1}{2}\int (1+x^2)' \cdot x^2\sqrt{1+x^2}\,dx$$

$$= \frac{1}{2}\int (1+x^2)'\left\{(1+x^2) - 1\right\}\sqrt{1+x^2}\,dx$$

$$= \frac{1}{2}\int (1+x^2)'\left\{(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}\right\}dx$$

$$= \frac{1}{2}\left\{\frac{2}{5}(1+x^2)^{\frac{5}{2}} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}\right\}$$

$$= \frac{1}{15}(3x^2-2)(x^2+1)\sqrt{x^2+1}.$$

(5)　$\sin x = t$ とおくと，$\cos x\,dx = dt$．よって

$$\int \cos^3 x \sin^2 x\,dx = \int \cos x(1-\sin^2 x)\sin^2 x\,dx = \int (1-t^2)t^2 dt = \int (t^2 - t^4)\,dt$$

$$= \frac{1}{3}t^3 - \frac{1}{5}t^5 = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x.$$

参考.
$$\int \cos^3 x \sin^2 x\,dx = \int \cos x(1-\sin^2 x)\sin^2 x\,dx$$

$$= \int (\sin x)'\left(\sin^2 x - \sin^4 x\right)dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x.$$

(6)　$\sqrt{2x-3} = t$ とおくと，$x = \frac{1}{2}(t^2+3)$，$dx = t\,dt$．よって

$$\int (x+1)\sqrt{2x-3}\,dx = \frac{1}{2}\int (t^2+5)t \cdot t\,dt = \frac{1}{2}\int (t^4 + 5t^2)\,dt$$

$$= \frac{1}{2}\left(\frac{1}{5}t^5 + \frac{5}{3}t^3\right) = \frac{1}{30}(3t^2+25)t^3$$
$$= \frac{1}{15}(3x+8)(2x-3)^{\frac{3}{2}}.$$

参考. $x+1 = \dfrac{1}{2}\Big\{(2x-3)+5\Big\}$ より,

$$\begin{aligned}\int (x+1)\sqrt{2x-3}\,dx &= \frac{1}{2}\int \{(2x-3)+5\}\sqrt{2x-3}\,dx \\ &= \frac{1}{2}\int \left\{(2x-3)^{\frac{3}{2}} + 5(2x-3)^{\frac{1}{2}}\right\}dx \\ &= \frac{1}{2}\left\{\frac{2}{5}\cdot\frac{1}{2}(2x-3)^{\frac{5}{2}} + 5\cdot\frac{2}{3}\cdot\frac{1}{2}(2x-3)^{\frac{3}{2}}\right\} \\ &= \frac{1}{10}(2-3)^{\frac{5}{2}} + \frac{5}{6}(2x-3)^{\frac{3}{2}} \\ &= \frac{1}{15}(3x+8)(2x-3)^{\frac{3}{2}}.\end{aligned}$$

(7) $\log x = t$ とおくと, $\frac{dx}{x} = dt$. よって

$$\int \frac{(\log x)^2}{x}\,dx = \int t^2\,dt = \frac{1}{3}t^3 = \frac{1}{3}(\log x)^3.$$

参考. $\displaystyle\int \frac{(\log x)^2}{x}\,dx = \int (\log x)'(\log x)^2\,dx = \frac{1}{3}(\log x)^3$

(8) $x = \tan t$ $\left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$ とおくと, $dx = \dfrac{dt}{\cos^2 t}$, $\cos t > 0$. よって

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{1}{(1+\tan^2 t)^{\frac{3}{2}}}\cdot\frac{dt}{\cos^2 t} = \int \cos^3 t\cdot\frac{dt}{\cos^2 t} = \int \cos t\,dt$$
$$= \sin t = \tan t\cos t = \frac{\tan t}{\sqrt{1+\tan^2 t}} = \frac{x}{\sqrt{1+x^2}}.$$

(9) $\dfrac{1}{x^2} = t$ とおくと, $x = \pm\dfrac{1}{\sqrt{t}}$. $x > 1$ のときは, $x = \dfrac{1}{\sqrt{t}}$ なので, $dx = -\dfrac{1}{2}t^{-\frac{3}{2}}dt$. よって

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{t}{\sqrt{\frac{1}{t}-1}}\left(-\frac{1}{2t\sqrt{t}}\right)dt = -\frac{1}{2}\int \frac{dt}{\sqrt{1-t}} = \sqrt{1-t}$$
$$= \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{\sqrt{x^2}} = \frac{\sqrt{x^2-1}}{x}.$$

$x < -1$ のときは, $x = -\dfrac{1}{\sqrt{t}}$ なので, $dx = \dfrac{1}{2}t^{-\frac{3}{2}}dt$. よって

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{t}{\sqrt{\frac{1}{t}-1}}\cdot\frac{dt}{2t\sqrt{t}} = \frac{1}{2}\int \frac{dt}{\sqrt{1-t}} = -\sqrt{1-t}$$
$$= -\sqrt{1-\frac{1}{x^2}} = -\frac{\sqrt{x^2-1}}{\sqrt{x^2}} = -\frac{\sqrt{x^2-1}}{-x} = \sqrt{1-\frac{1}{x^2}}.$$

以上より

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \sqrt{1-\frac{1}{x^2}}.$$

参考.　$\sqrt{x^2-1}=t-x$ とおくと，

$$x=\frac{t^2+1}{2t},\quad dx=\frac{t^2-1}{2t^2}\,dt,\quad \sqrt{x^2-1}=\frac{t^2-1}{2t}.$$

$$\begin{aligned}\therefore\quad \int\frac{dx}{x^2\sqrt{x^2-1}}&=\int\frac{4t^2}{(t^2+1)^2}\cdot\frac{2t}{t^2-1}\cdot\frac{t^2-1}{2t^2}\,dt=4\int\frac{t}{(t^2+1)^2}\,dt\\&=-\frac{2}{t^2+1}=-\frac{2}{(\sqrt{x^2-1}+x)^2+1}\\&=-\frac{1}{x(\sqrt{x^2-1}+x)}=\frac{\sqrt{x^2-1}-x}{x}=\frac{\sqrt{x^2-1}}{x}-1.\end{aligned}$$

参考.　$\sqrt{x^2-1}=t+x$ とおくと，

$$x=-\frac{t^2+1}{2t},\quad dx=-\frac{t^2-1}{2t^2}\,dt,\quad \sqrt{x^2-1}=\frac{t^2-1}{2t}.$$

$$\begin{aligned}\therefore\quad \int\frac{dx}{x^2\sqrt{x^2-1}}&=\int\frac{4t^2}{(t^2+1)^2}\cdot\frac{2t}{t^2-1}\cdot\left(-\frac{t^2-1}{2t^2}\right)dt=-4\int\frac{t}{(t^2+1)^2}\,dt\\&=\frac{2}{t^2+1}=\frac{2}{(\sqrt{x^2-1}-x)^2+1}\\&=\frac{1}{x(x-\sqrt{x^2-1})}=\frac{x+\sqrt{x^2-1}}{x}=\frac{\sqrt{x^2-1}}{x}+1.\end{aligned}$$

(10)　$\sin^{-1}x=t$ とおくと，$\dfrac{dx}{\sqrt{1-x^2}}=dt$．よって

$$\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx=\int t\,dt=\frac{1}{2}t^2=\frac{1}{2}(\sin^{-1}x)^2.$$

参考.　$\displaystyle\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx=\int(\sin^{-1}x)'\sin^{-1}x\,dx=\frac{1}{2}(\sin^{-1}x)^2$

問 4.　(1)　$\displaystyle\int x^2e^x\,dx=x^2e^x-\int 2xe^x\,dx=x^2e^x-2xe^x+2\int e^x\,dx=(x^2-2x+2)e^x$

(2)　$\displaystyle\int(x-2)\cos 3x\,dx=\frac{1}{3}(x-2)\sin 3x-\frac{1}{3}\int\sin 3x\,dx=\frac{1}{3}(x-2)\sin 3x+\frac{1}{9}\cos 3x$

(3)　$\displaystyle I=\int e^x\cos x\,dx$ とおくと，

$$I=e^x\cos x+\int e^x\sin x\,dx=e^x(\cos x+\sin x)-\int e^x\cos x\,dx=e^x(\cos x+\sin x)-I$$

$$\therefore\quad\int e^x\cos x\,dx=\frac{1}{2}e^x(\sin x+\cos x).$$

(4)　$\displaystyle\int\frac{\log x}{x^2}\,dx=-\frac{\log x}{x}+\int\frac{dx}{x^2}=-\frac{\log x}{x}-\frac{1}{x}=-\frac{\log x+1}{x}$

(5)　$\displaystyle\int\log(x^2+1)\,dx=\int x'\log(x^2+1)\,dx=x\log(x^2+1)-\int\frac{2x^2}{x^2+1}\,dx$

$$= x\log(x^2+1) - 2\int \frac{(x^2+1)-1}{x^2+1}\,dx$$
$$= x\log(x^2+1) - 2\int \left(1 - \frac{1}{x^2+1}\right)dx$$
$$= x\log(x^2+1) + 2\tan^{-1}x - 2x$$

(6) $\displaystyle\int (\log x)^2\,dx = \int x'(\log x)^2\,dx = x(\log x)^2 - \int x\cdot\frac{2}{x}\log x\,dx = x(\log x)^2 - 2\int \log x\,dx$

$$= x(\log x)^2 - 2x\log x + 2x = x\Big\{(\log x)^2 - 2\log x + 2\Big\}$$

(7) $\displaystyle\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x$$

(8) $\displaystyle\int \sin^{-1}x\,dx = \int x'\sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x + \sqrt{1-x^2}$

(9) $\displaystyle\int \tan^{-1}x\,dx = \int x'\tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{x^2+1}\,dx = x\tan^{-1}x - \frac{1}{2}\log(x^2+1)$

(10) $\displaystyle I = \int \sqrt{x^2+2x+2}\,dx$ とおくと,

$$I = \int (x+1)'\sqrt{x^2+2x+2}\,dx = (x+1)\sqrt{x^2+2x+2} - \int \frac{(x+1)^2}{\sqrt{x^2+2x+2}}\,dx$$
$$= (x+1)\sqrt{x^2+2x+2} - \int \frac{(x^2+2x+2)-1}{\sqrt{x^2+2x+2}}\,dx$$
$$= (x+1)\sqrt{x^2+2x+2} - \int \sqrt{x^2+2x+2}\,dx + \int \frac{dx}{\sqrt{(x+1)^2+1}}$$
$$= (x+1)\sqrt{x^2+2x+2} - I + \log\left|x+1+\sqrt{(x+1)^2+1}\right|.$$

上式を I について解いて,

$$\int \sqrt{x^2+2x+2}\,dx = \frac{1}{2}(x+1)\sqrt{x^2+2x+2} + \frac{1}{2}\log\left(x+1+\sqrt{x^2+2x+2}\right).$$

(11) $\displaystyle\int \sqrt{1-4x-x^2}\,dx = \int \sqrt{5-(x+2)^2}\,dx = \frac{1}{2}(x+2)\sqrt{5-(x+2)^2} + \frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}$

$$= \frac{1}{2}(x+2)\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}.$$

(12) $\displaystyle\int \frac{x^2}{\sqrt{x^2+2}}\,dx = \int x\cdot\frac{x}{\sqrt{x^2+1}}\,dx = x\sqrt{x^2+1} - \int \sqrt{x^2+2}\,dx$

$$= x\sqrt{x^2+2} - \frac{1}{2}\left(x\sqrt{x^2+2} + 2\log\left|x+\sqrt{x^2+2}\right|\right)$$
$$= \frac{1}{2}x\sqrt{x^2+2} - \log\left(x+\sqrt{x^2+2}\right)$$

問 5.　I と J をそれぞれ部分積分すると，

$$I = \int e^{ax} \sin bx \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} J$$

$$\therefore \quad bI - aJ = -e^{ax} \cos bx \tag{1}$$

$$J = \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I$$

$$\therefore \quad aI + bJ = e^{ax} \sin bx \tag{2}$$

$(2) \times a + (1) \times b$ より，

$$(a^2 + b^2) I = e^{ax} (a \sin bx - b \cos bx).$$

また，$(2) \times b - (1) \times a$ より，

$$(a^2 + b^2) J = e^{ax} (b \sin bx + a \cos bx).$$

ゆえに

$$I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx), \quad J = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx).$$

問 6.　部分積分法を用いて計算すると，

$$\begin{aligned} A\, I_n &= \int \frac{A}{(x^2 + A)^n} \, dx = \int \frac{(x^2 + A) - x^2}{(x^2 + A)^n} \, dx \\ &= \int \frac{dx}{(x^2 + A)^{n-1}} - \int \frac{x}{(x^2 + A)^n} \cdot x \, dx \\ &= I_{n-1} + \frac{1}{2(n-1)} \left\{ \frac{1}{(x^2 + A)^{n-1}} \cdot x - \int \frac{dx}{(x^2 + A)^{n-1}} \right\} \\ &= \frac{1}{2(n-1)} \left\{ \frac{x}{(x^2 + A)^{n-1}} + (2n-3) I_{n-1} \right\}. \end{aligned}$$

よって

$$I_n = \frac{1}{2(n-1)A} \left\{ \frac{x}{(x^2 + A)^{n-1}} + (2n-3) I_{n-1} \right\} \quad (n \geqq 2)$$

次に，上で示した漸化式より，

$$\begin{aligned} \int \frac{dx}{(x^2 + 1)^2} = I_2 = \frac{1}{2} \left(\frac{x}{x^2 + 1} + I_1 \right) &= \frac{1}{2} \left(\frac{x}{x^2 + 1} + \int \frac{dx}{x^2 + 1} \right) \\ &= \frac{1}{2} \left(\frac{x}{x^2 + 1} + \tan^{-1} x \right). \end{aligned}$$

参考.　部分積分法を用いて計算すると，

$$\begin{aligned} \int \frac{dx}{x^2 + 1} &= \int x' \frac{dx}{x^2 + 1} = \frac{x}{x^2 + 1} - \int x \left(\frac{1}{x^2 + 1} \right)' dx \\ &= \frac{x}{x^2 + 1} + \int \frac{2x^2}{(x^2 + 1)^2} \, dx = \frac{x}{x^2 + 1} + \int \frac{2(x^2 + 1) - 2}{(x^2 + 1)^2} \, dx \\ &= \frac{x}{x^2 + 1} + 2 \int \frac{dx}{x^2 + 1} - 2 \int \frac{dx}{(x^2 + 1)^2}. \end{aligned}$$

$$\therefore \quad \int \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \left(\frac{x}{x^2 + 1} + \int \frac{dx}{x^2 + 1} \right) = \frac{1}{2} \left(\frac{x}{x^2 + 1} + \tan^{-1} x \right).$$

参考． $n=1$ のときを考える．$A>0$ のときは，

$$I_1=\int\frac{dx}{x^2+(\sqrt{A})^2}=\frac{1}{\sqrt{A}}\tan^{-1}\frac{x}{\sqrt{A}}.$$

$A<0$ のときは，

$$\begin{aligned}I_1&=\int\frac{dx}{x^2-(\sqrt{-A})^2}=\frac{1}{2\sqrt{-A}}\int\left(\frac{1}{x-\sqrt{-A}}-\frac{1}{x+\sqrt{-A}}\right)dx\\&=\frac{1}{2\sqrt{-A}}\log\left|\frac{x-\sqrt{-A}}{x+\sqrt{-A}}\right|.\end{aligned}$$

$$\therefore\quad I_1=\begin{cases}\dfrac{1}{\sqrt{A}}\tan^{-1}\dfrac{x}{\sqrt{A}} & (A>0)\\[2ex] \dfrac{1}{2\sqrt{-A}}\log\left|\dfrac{x-\sqrt{-A}}{x+\sqrt{-A}}\right| & (A<0).\end{cases}$$

問 7. (1)

$$\begin{aligned}\int\frac{dx}{(x+1)(x+3)}&=\frac{1}{2}\int\left(\frac{1}{x+1}-\frac{1}{x+3}\right)dx=\frac{1}{2}\left(\log|x+1|-\log|x+3|\right)\\&=\frac{1}{2}\log\left|\frac{x+1}{x+3}\right|\end{aligned}$$

(2) $\dfrac{3x^2-x+1}{x(x-1)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{(x-1)^2}$ とおくと，

$$3x^2-x+1=(A+B)x^2+(-2A+C)x+A.$$

上式の両辺の係数を比較すると， $A+B=3$，$-2A+C=-1$，$A=1$． よって，$A=C=1$，$B=2$．ゆえに

$$\frac{3x^2-x+1}{x(x-1)^2}=\frac{1}{x}+\frac{2x+1}{(x-1)^2}=\frac{1}{x}+\frac{2(x-1)+3}{(x-1)^2}=\frac{1}{x}+\frac{2}{x-1}+\frac{3}{(x-1)^2}.$$

よって

$$\begin{aligned}\int\frac{3x^2-x+1}{x(x-1)^2}\,dx&=\int\left\{\frac{1}{x}+\frac{2}{x-1}+\frac{3}{(x-1)^2}\right\}dx\\&=\log|x|+2\log|x-1|-\frac{3}{x-1}\\&=\log\left|x(x-1)^2\right|-\frac{3}{x-1}.\end{aligned}$$

(3) $\dfrac{1}{x^3+1}=\dfrac{1}{(x+1)(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}$ とおくと，

$$1=(A+B)x^2+(-A+B+C)x+A+C.$$

上式の両辺の係数を比較すると，$A+B=0$，$-A+B+C=0$，$A+C=1$．よって，$A=-B=\dfrac{1}{3}$，$C=\dfrac{2}{3}$．ゆえに

$$\frac{1}{x^3+1}=\frac{1}{3}\left(\frac{1}{x+1}-\frac{x-2}{x^2-x+1}\right).$$

よって

$$\int\frac{dx}{x^3+1}=\frac{1}{3}\int\left(\frac{1}{x+1}-\frac{1}{2}\cdot\frac{(2x-1)-3}{x^2-x+1}\right)dx$$

$$= \frac{1}{6}\int\left(\frac{2}{x+1}-\frac{2x-1}{x^2-x+1}+\frac{3}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)dx$$

$$= \frac{1}{6}\left\{2\log|x+1|-\log(x^2-x+1)+3\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right\}$$

$$= \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

(4)　部分分数に分解すると，

$$\frac{1}{x^4-1}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{2}\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)$$

$$=\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)-\frac{1}{2(x^2+1)}.$$

よって

$$\int\frac{dx}{x^4-1}=\frac{1}{4}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx-\frac{1}{2}\int\frac{dx}{x^2+1}$$

$$=\frac{1}{4}(\log|x-1|-\log|x+1|)-\frac{1}{2}\tan^{-1}x$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x.$$

(5)　部分分数に分解すると，

$$\frac{x+1}{x(x^2+1)}=\frac{1}{x}-\frac{x-1}{x^2-1}=\frac{1}{x}-\frac{1}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{x^2+1}.$$

よって

$$\int\frac{x+1}{x(x^2+1)}dx=\int\left(\frac{1}{x}-\frac{1}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$=\log|x|-\frac{1}{2}\log(x^2+1)+\tan^{-1}x$$

$$=\frac{1}{2}\log\frac{x^2}{x^2+1}+\tan^{-1}x.$$

(6)　$\dfrac{x-2}{(x-1)^2(x^2-x+1)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2-x+1}$ とおくと，

$$x-2=(A+C)x^3+(-2A+B-2C+D)x^2+(2A-B+C-2D)x+(-A+B+D).$$

上式の両辺の係数を比較すると，$A+C=0$，$-2A+B-2C+D=0$，$2A-B+C-2D=1$，$-A+B+D=-2$．よって， $A=2$，$B=-1$，$C=-2$，$D=1$．ゆえに

$$\frac{x-2}{(x-1)^2(x^2-x+1)}=\frac{2}{x-1}-\frac{1}{(x-1)^2}-\frac{2x-1}{x^2-x+1}.$$

ゆえに

$$\int\frac{x-2}{(x-1)^2(x^2-x+1)}dx=\int\left\{\frac{2}{x-1}-\frac{1}{(x-1)^2}-\frac{2x-1}{x^2-x+1}\right\}dx$$

$$=2\log|x-1|+\frac{1}{x-1}-\log(x^2-x+1)$$

$$= \log \frac{(x-1)^2}{x^2-x+1} + \frac{1}{x-1}.$$

(7) $\dfrac{x^4}{x^3+1} = x - \dfrac{x}{x^3+1}$. そこで,

$$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

とおくと,

$$x = (A+B)x^2 + (-A+B+C)x + A + C.$$

両辺の係数を比較すると，$A+B=0$，$-A+B+C=1$，$A+C=0$. よって， $A=-\dfrac{1}{3}$，$B=C=\dfrac{1}{3}$. ゆえに

$$\begin{aligned}\frac{x^4}{x^3+1} &= x - \frac{1}{3}\left(-\frac{1}{x+1} + \frac{x+1}{x^2-x+1}\right)\\ &= x + \frac{1}{3(x+1)} - \frac{1}{6}\left\{\frac{2x-1}{x^2-1+1} + \frac{3}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}.\end{aligned}$$

よって

$$\begin{aligned}\int \frac{x^4}{x^3+1} &= \int \left\{x + \frac{1}{6}\left(\frac{2}{x+1} - \frac{2x-1}{x^2-1+1}\right) - \frac{1}{2}\cdot\frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}dx\\ &= \frac{x^2}{2} + \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.\end{aligned}$$

(8) $\dfrac{x^3}{(x-1)(x-2)} = x+3+\dfrac{7x-6}{(x-1)(x-2)} = x+3-\dfrac{1}{x-1}+\dfrac{8}{x-2}$ なので,

$$\begin{aligned}\int \frac{x^3}{(x-1)(x-2)}\,dx &= \int\left(x+3-\frac{1}{x-1}+\frac{8}{x-2}\right)dx\\ &= \frac{x^2}{2} + 3x - \log|x-1| + 8\log|x-2|.\end{aligned}$$

(9) $\dfrac{x^5}{x^4+x^2-2} = x - \dfrac{x^3-2x}{x^4+x^2-2}$. そこで

$$\frac{x^3-2x}{x^4+x^2-2} = \frac{x^3-2x}{(x-1)(x+1)(x^2+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2}$$

とおいて部分分数に分解すると,

$$\frac{x^3-2x}{x^4+x^2-2} = -\frac{1}{6}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{1}{x+1} + \frac{4}{3}\cdot\frac{x}{x^2+2}.$$

よって

$$\int \frac{x^5}{x^4+x^2-2} = \int\left(x + \frac{1}{6}\cdot\frac{1}{x-1} + \frac{1}{6}\cdot\frac{1}{x+1} - \frac{4}{3}\cdot\frac{x}{x^2+2}\right)dx$$

$$= \frac{x^2}{2} + \frac{1}{6}\log|x-1| + \frac{1}{6}\log|x+1| - \frac{2}{3}\log(x^2+2)$$
$$= \frac{x^2}{2} + \frac{1}{6}\log|x^2-1| - \frac{2}{3}\log(x^2+2).$$

別解. $\displaystyle\int \frac{x^5}{x^4+x^2-2}\,dx = \frac{1}{2}\int \frac{x^4}{x^4+x^2-2}\cdot 2x\,dx$. $x^2 = t$ とおくと，$2x\,dx = dt$. よって

$$\int \frac{x^5}{x^4+x^2-2}\,dx = \frac{1}{2}\int \frac{t^2}{t^2+t-2}\,dt = \frac{1}{2}\int \left\{1 - \frac{t-2}{(t-1)(t+2)}\right\}dt$$
$$= \frac{1}{2}\int \left\{1 + \frac{1}{3}\left(\frac{1}{t-1} - \frac{4}{t+2}\right)\right\}dt$$
$$= \frac{t}{2} + \frac{1}{6}\big(\log|t-1| - 4\log|t+2|\big)$$
$$= \frac{x^2}{2} + \frac{1}{6}\log|x^2-1| - \frac{2}{3}\log(x^2+2).$$

(10)　部分分数に分解すると，

$$\frac{1}{x^4+3x^2-4} = \frac{1}{(x^2-1)(x^2+4)} = \frac{1}{5}\left(\frac{1}{x^2-1} - \frac{1}{x^2+4}\right)$$
$$= \frac{1}{10}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) - \frac{1}{5(x^2+4)}.$$

よって

$$\int \frac{dx}{x^4+3x^2-4} = \frac{1}{10}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx - \frac{1}{5}\int\frac{dx}{x^2+4}$$
$$= \frac{1}{10}(\log|x-1| - \log|x+1|) - \frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}$$
$$= \frac{1}{10}\left(\log\left|\frac{x-1}{x+1}\right| - \tan^{-1}\frac{x}{2}\right).$$

(11)　$e^x = X$ とおいて，X に関して部分分数に分解すると，

$$\frac{e^x}{e^{2x}-1} = \frac{X}{X^2-1} = \frac{1}{2}\left(\frac{X}{X-1} - \frac{X}{X+1}\right) = \frac{1}{2}\left(\frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}\right).$$

よって

$$\int \frac{e^x}{e^{2x}-1}\,dx = \frac{1}{2}\int\left(\frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}\right)dx = \frac{1}{2}\left(\log|e^x-1| - \log|e^x+1|\right)$$
$$= \frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right| = \frac{1}{2}\log\left|\tanh\frac{x}{2}\right|.$$

(12)　$X = e^x$ とおいて，以下のように変形する．

$$\frac{1}{e^x+4e^{-x}+5} = \frac{1}{X+\dfrac{4}{X}+5} = \frac{X}{X^2+5X+4} = \frac{X}{(X+1)(X+4)}$$

$$= \frac{1}{3}\left(\frac{X}{X+1} - \frac{X}{X+4}\right) = \frac{1}{3}\left(\frac{e^x}{e^x+1} - \frac{e^x}{e^x+4}\right).$$

よって

$$\int \frac{dx}{e^x+4e^{-x}+5} = \frac{1}{3}\int\left(\frac{e^x}{e^x+1} - \frac{e^x}{e^x+4}\right)d = \frac{1}{3}\left(\log|e^x+1| - \log|e^x+4|\right)$$
$$= \frac{1}{3}\log\frac{e^x+1}{e^x+4}.$$

別解.　$e^x = t$ とおくと，$e^x\,dx = dt$，$e^x + 4e^{-x} + 5 = t + \frac{4}{t} + 5 = \frac{t^2+5t+4}{t}$．よって

$$\int \frac{dx}{e^x+4^{-x}+5} = \int \frac{t}{t^2+5t+4}\cdot\frac{dt}{t} = \int \frac{dt}{(t+1)(t+4)}$$
$$= \frac{1}{3}\int\left(\frac{1}{t+1} - \frac{1}{t+4}\right)dt = \frac{1}{3}\left(\log|t+1| - \log|t+4|\right)$$
$$= \frac{1}{3}\log\left|\frac{t+1}{t+4}\right| = \frac{1}{3}\log\frac{e^x+1}{e^x+4}.$$

問 8.　(1)　$a>0$ のとき，$\sqrt{ax^2+bx+c} = t - \sqrt{a}x$ とおくと，

$$x = \frac{t^2-c}{2\sqrt{a}t+b},\quad dx = \frac{2\left(\sqrt{a}t^2+bt+\sqrt{a}c\right)}{(2\sqrt{a}t+b)^2}\,dt,\quad \sqrt{ax^2+bx+c} = \frac{\sqrt{a}t^2+bt+\sqrt{a}c}{2\sqrt{a}t+b}.$$

よって

$$I = \int R\left(\frac{t^2-c}{2\sqrt{a}t+b}, \frac{\sqrt{a}t^2+bt+\sqrt{a}c}{2\sqrt{a}t+b}\right)\cdot\frac{2\left(\sqrt{a}t^2+bt+\sqrt{a}c\right)}{(2\sqrt{a}t+b)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

(2)　$a<0$ のとき，$\sqrt{\frac{x-\alpha}{\beta-x}} = t$ とおくと，

$$x = \frac{\beta t^2+\alpha}{t^2+1},\quad dx = \frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となる．また，解と係数の関係

$$\alpha+\beta = -\frac{b}{a},\quad \alpha\beta = \frac{c}{a}$$

を用いれば,

$$ax^2+bx+c = \frac{\{2a\alpha\beta + b(\alpha+\beta) + 2c\}t^2}{(t^2+1)^2} = \frac{(b^2-4ac)t^2}{-a(t^2+1)^2}$$
$$\therefore\quad \sqrt{ax^2+bx+c} = \sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}$$

を得る．よって

$$I = \int R\left(\frac{\beta t^2+\alpha}{t^2+1}, \sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}\right)\cdot\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

一方，$a<0$ のとき，$\sqrt{\frac{\beta-x}{x-\alpha}} = t$ とおくと，

$$x = \frac{\alpha t^2+\beta}{t^2+1},\quad dx = -\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となる．また，

$$ax^2+bx+c=\frac{\{2a\alpha\beta+b(\alpha+\beta)+2c\}t^2}{(t^2+1)^2}=\frac{(b^2-4ac)t^2}{-a(t^2+1)^2}$$

$$\therefore\quad \sqrt{ax^2+bx+c}=\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}.$$

よって

$$I=-\int R\left(\frac{\alpha t^2+\beta}{t^2+1},\sqrt{\frac{b^2-4ac}{-a}}\cdot\frac{t}{t^2+1}\right)\cdot\frac{2(\beta-\alpha)t}{(t^2+1)^2}\,dt$$

となり，t に関する有理関数の積分に帰着できる．

参考． $a>0$ のとき，$\sqrt{ax^2+bx+c}=t+\sqrt{a}x$ とおくと，

$$x=\frac{-t^2+c}{2\sqrt{a}t-b},\quad dx=-\frac{2\left(\sqrt{a}t^2-bt+\sqrt{a}c\right)}{(2\sqrt{a}t-b)^2}\,dt,\quad \sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2-bt+\sqrt{a}c}{2\sqrt{a}t-b}$$

より

$$I=-\int R\left(\frac{-t^2+c}{2\sqrt{a}t-b},\frac{\sqrt{a}t^2-bt+\sqrt{a}c}{2\sqrt{a}t-b}\right)\cdot\frac{2\left(\sqrt{a}t^2-bt+\sqrt{a}c\right)}{(2\sqrt{a}t-b)^2}\,dt$$

となり，やはり t に関する有理関数の積分に帰着できる．

問 9. (1) $\sqrt{1-x}=t$ とおくと，$x=1-t^2$，$dx=-2t\,dt$．よって

$$\int\frac{dx}{x\sqrt{1-x}}=\int\frac{-2t}{t(1-t^2)}\,dt=\int\frac{2}{t^2-1}\,dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$
$$=\log|t-1|-\log|t+1|=\log\left|\frac{t-1}{t+1}\right|$$
$$=\log\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right|.$$

(2) $\sqrt{\dfrac{x-1}{x+1}}=t$ とおくと，$x=-\dfrac{t^2+1}{t^2-1}$，$dx=\dfrac{4t}{(t^2-1)^2}\,dt$．よって

$$\int\sqrt{\frac{x-1}{x+1}}\,dx=\int t\cdot\frac{4t}{(t^2-1)^2}\,dt=4\int\frac{(t^2-1)+1}{(t^2-1)^2}\,dt=4\int\left\{\frac{1}{t^2-1}+\frac{1}{(t^2-1)^2}\right\}dt$$

ここで

$$\int\frac{dt}{t^2-1}=\int t'\cdot\frac{1}{t^2-1}\,dt=\frac{t}{t^2-1}+\int\frac{2t^2}{(t^2-1)^2}\,dt$$
$$=\frac{t}{t^2-1}+2\int\frac{(t^2-1)+1}{(t^2-1)^2}\,dt$$
$$=\frac{t}{t^2-1}+2\int\frac{dt}{t^2-1}+2\int\frac{dt}{(t^2-1)^2}$$

より

$$\int\frac{dt}{(t^2-1)^2}=-\frac{1}{2}\left(\frac{t}{t^2-1}+\int\frac{dt}{t^2-1}\right).$$

よって

$$\int\sqrt{\frac{x-1}{x+1}}\,dx=4\int\frac{dt}{t^2-1}-2\left(\frac{t}{t^2-1}+\int\frac{dt}{t^2-1}\right)$$

$$= -\frac{2t}{t^2-1} + 2\int \frac{dt}{t^2-1} = -\frac{2t}{t^2-1} + \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= -\frac{2t}{t^2-1} + \log\left|\frac{t-1}{t+1}\right| = -\frac{2\sqrt{\frac{x-1}{x+1}}}{\frac{x-1}{x+1}-1} + \log\left|\frac{\sqrt{\frac{x-1}{x+1}}-1}{\sqrt{\frac{x-1}{x+1}}+1}\right|$$

$$= \sqrt{x^2-1} + \log\left|\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\right|$$

$$= \sqrt{x^2-1} + \log\left|\frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}{x+1-(x-1)}\right|$$

$$= \sqrt{x^2-1} + \log\left|x - \sqrt{x^2-1}\right|.$$

(3)　$\sqrt{x^2+x+1} = t - x$ とおくと，

$$x = \frac{t^2-1}{2t+1},\quad dx = \frac{2(t^2+t+1)}{(2t+1)^2}\,dt,\quad \sqrt{x^2+x+1} = \frac{t^2+t+1}{2t+1}.$$

よって

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{2t+1}{t^2-1}\cdot\frac{2t+1}{t^2+t+1}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt$$

$$= \int \frac{2}{(t-1)(t+1)}\,dt = \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= \log\left|\frac{t-1}{t+1}\right| = \log\left|\frac{\sqrt{x^2+x+1}+x-1}{\sqrt{x^2+x+1}+x+1}\right|.$$

参考.　$\sqrt{x^2+x+1} = t + x$ とおくと，

$$x = -\frac{t^2-1}{2t-1},\quad dx = -\frac{2(t^2-t+1)}{(2t-1)^2}\,dt,\quad \sqrt{x^2+x+1} = \frac{t^2-t+1}{2t-1}.$$

よって

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{2t-1}{t^2-1}\cdot\frac{2t-1}{t^2-t+1}\cdot\frac{2(t^2-t+1)}{(2t-1)^2}\,dt$$

$$= \int \frac{2}{t^2-1}\,dt = \log\left|\frac{t-1}{t+1}\right|$$

$$= \log\left|\frac{\sqrt{x^2+x+1}-x-1}{\sqrt{x^2+x+1}-x+1}\right|.$$

(4)　$\sqrt{4x-3-x^2} = \sqrt{(x-1)(3-x)} = (3-x)\sqrt{\frac{x-1}{3-x}}$ より

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{dx}{x(3-x)\sqrt{\frac{x-1}{3-x}}}.$$

そこで，$\sqrt{\frac{x-1}{3-x}} = t$ とおくと，$x = \frac{3t^2+1}{t^2+1}$，$dx = \frac{4t}{(t^2+1)^2}\,dt$，$3-x = \frac{2}{t^2+1}$. よって

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{t^2+1}{3t^2+1}\cdot\frac{t^2+1}{2}\cdot\frac{1}{t}\,\frac{4t}{(t^2+1)^2}\,dt = 2\int \frac{dt}{3t^2+1}$$

$$= \frac{2}{3}\int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{3}\cdot\sqrt{3}\tan^{-1}\sqrt{3}t$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3(x-1)}{3-x}}.$$

参考. $\sqrt{4x-3-x^2} = \sqrt{(x-1)(3-x)} = (x-1)\sqrt{\frac{3-x}{x-1}}$ より

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{dx}{x(x-1)\sqrt{\frac{3-x}{x-1}}}.$$

そこで，$\sqrt{\frac{3-x}{x-1}} = t$ とおくと，$x = \frac{t^2+3}{t^2+1}$，$dx = -\frac{4t}{(t^2+1)^2}\,dt$，$x-1 = \frac{2}{t^2+1}$．よって

$$\int \frac{dx}{x\sqrt{4x-3-x^2}} = \int \frac{t^2+1}{t^2+3}\cdot\frac{t^2+1}{2}\cdot\frac{1}{t}\,\frac{-4t}{(t^2+1)^2}\,dt = -2\int \frac{dt}{t^2+3}$$

$$= -\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} = -\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3-x}{3(x-1)}}.$$

参考. $x > 0$ のとき，$\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$ である.

(5) $\tan\frac{x}{2} = t$ とおくと，$dx = \frac{2}{t^2+1}\,dt$，$\cos x = \frac{1-t^2}{1+t^2}$．よって

$$\int \frac{dx}{\cos x} = \int \frac{1+t^2}{1-t^2}\cdot\frac{2}{t^2+1}\,dt = \int \frac{2}{1-t^2}\,dt = \int\left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt$$

$$= \log\left|\frac{1+t}{1-t}\right| = \log\left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}-1}\right| = \log\left|\frac{1+\sin x}{\cos x}\right|.$$

参考. $\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{\cos x}{(1-\sin x)(1+\sin x)}\,dx$

$$= \frac{1}{2}\int\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)dx$$

$$= \frac{1}{2}\int\left\{-\frac{(1-\sin x)'}{1-\sin x} + \frac{(1+\sin x)'}{1+\sin x}\right\}dx$$

$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

(6) $\tan\frac{x}{2} = t$ とおくと，$dx = \frac{2}{t^2+1}\,dt$，$\sin x = \frac{2}{t^2+1}$．よって

$$\int \frac{\sin x}{1+\sin x}\,dx = \int\left(1 - \frac{1}{1+\sin x}\right)dx = x - \int \frac{dx}{1+\sin x}$$

$$= x - \int \frac{t^2+1}{(t+1)^2}\cdot\frac{2}{t^2+1}\,dt = x - 2\int \frac{dt}{(t+1)^2}$$

$$= x + \frac{2}{t+1} = x + \frac{2}{\tan\frac{x}{2}+1}.$$

参考. $\displaystyle\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\,dx$

$$= \int \frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\,dx = \int \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} + 1\right)dx$$

$$= \frac{1}{\cos x} - \tan x + x = x + \frac{1-\sin x}{\cos x}.$$

問 10. (1) $e^x = t$ とおくと，$dx = \frac{dt}{t}$．よって

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{t^2-1}{t(t^2+1)}\,dt = \int \left(\frac{2t}{t^2+1} - \frac{1}{t}\right)dt$$

$$= \log(t^2+1) - \log|t| = \log\frac{t^2+1}{t}$$

$$= \log(e^x + e^{-x}).$$

参考. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{(e^x + e^{-x})'}{e^x + e^{-x}}\,dx = \log(e^x + e^{-x})$

参考. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \tanh x\,dx = \log\cosh x = \log\frac{e^x + e^{-x}}{2} = \log(e^e + e^{-x}) - \log 2.$

(2) $\log x = t$ とおくと，$\dfrac{dx}{x} = dt$．よって

$$\int \frac{(\log x + 3)^5}{x}\,dx = \int (t+3)^5\,dt = \frac{1}{6}(t+3)^6 = \frac{1}{6}(\log x + 3)^6.$$

参考. $\displaystyle\int \frac{(\log x + 3)^5}{x}\,dx = \int (\log x + 3)'(\log x + 3)^5\,dx = \frac{1}{6}(\log x + 3)^6$

(3) $\sin x = t$ $\left(-\dfrac{\pi}{2} \leqq x \leqq \dfrac{\pi}{2}\right)$ とおくと，$\cos x\,dx = dt$．よって

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1}(\sin x).$$

(4) $\cos x = t$ $(0 \leqq x \leqq \pi)$ とおくと，$\sin x\,dx = -dt$．よって

$$\int \frac{\cos^2 x \sin x}{3\cos^2 x + \sin^2 x}\,dx = \int \frac{\cos^2 x \sin x}{1 + 2\cos^2 x}\,dx = \int \frac{-t^2}{1+2t^2}\,dt$$

$$= -\frac{1}{2}\int \left(1 - \frac{\frac{1}{2}}{t^2 + \frac{1}{2}}\right)dt$$

$$= -\frac{t}{2} + \frac{1}{4}\cdot\sqrt{2}\tan^{-1}\left(\sqrt{2}t\right)$$

$$= -\frac{1}{2}\cos x + \frac{\sqrt{2}}{4}\tan^{-1}\left(\sqrt{2}\cos x\right).$$

(5) $\tan x = t$ $\left(-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right)$ とおくと，$dx = \dfrac{dt}{1+t^2}$．よって

$$\int \frac{dx}{1+\tan x} = \int \frac{1}{1+t}\cdot\frac{1}{1+t^2}\,dt = \frac{1}{2}\int \left(\frac{1}{1+t} + \frac{1-t}{1+t^2}\right)dt$$

$$= \frac{1}{2}\int \left(\frac{1}{1+t} - \frac{t}{1+t^2} + \frac{1}{1+t^2}\right)dt$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{\log|1+t| - \frac{1}{2}\log(1+t^2) + \tan^{-1} t\right\} \\
&= \frac{1}{4}\log\frac{(1+\tan x)^2}{1+\tan^2 x} + \frac{x}{2} = \frac{1}{4}\log(\sin x + \cos x)^2 + \frac{x}{2} \\
&= \frac{1}{4}\log(1+\sin 2x) + \frac{x}{2}.
\end{aligned}
$$

(6)　$\tan x = t \ \left(-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right)$ とおくと，$dx = \dfrac{dt}{1+t^2}$．よって

$$
\begin{aligned}
\int \frac{\tan^2 x}{3+\tan^3 x}\,dx &= \int \frac{t^2}{3+t^2}\cdot\frac{1}{1+t^2}\,dt = \frac{1}{2}\int\left(\frac{3}{3+t^2} - \frac{1}{1+t^2}\right)dt \\
&= \frac{3}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} - \frac{1}{2}\tan^{-1} t \\
&= \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right) - \frac{x}{2}.
\end{aligned}
$$