第 2 章の解答例

2.1 導関数

問 1.

(2) $\left(\sqrt{x}\right)' = \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \dfrac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$

$= \lim_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}.$

(4) $(\log x)' = \lim_{h \to 0} \dfrac{\log(x+h) - \log x}{h} = \lim_{h \to 0} \dfrac{\log \frac{x+h}{x}}{h} = \lim_{h \to 0} \dfrac{1}{h} \log\left(1 + \dfrac{h}{x}\right)$

$= \dfrac{1}{x} \cdot \lim_{h \to 0} \dfrac{1}{\frac{h}{x}} \log\left(1 + \dfrac{h}{x}\right) = \dfrac{1}{x}.$

(6) $(\cos x)' = \lim_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \dfrac{-2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{h}$

$= -\lim_{h \to 0} \sin\left(x + \dfrac{h}{2}\right) \dfrac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin x.$

問 2.

(1) $(x^4 - 3x + 1)' = 4x^3 - 3.$

(2) $\{(x-1)(x^2+1)\}' = (x^2+1) + (x-1) \cdot 2x = 3x^2 - 2x + 1.$

(3) $\left(\dfrac{x}{x-1}\right)' = \dfrac{x - 1 - x}{(x-1)^2} = -\dfrac{1}{(x-1)^2}.$

(4) $\left(\dfrac{x}{x^2+1}\right)' = \dfrac{x^2 + 1 - x \cdot 2x}{(x^2+1)^2} = -\dfrac{x^2 - 1}{(x^2+1)^2}.$

(5) $\left(\dfrac{x^2+x+1}{x^2-x+1}\right)' = \dfrac{(2x+1)(x^2-x+1) - (x^2+x+1)(2x-1)}{(x^2-x+1)^2} = \dfrac{-2(x^2-1)}{(x^2-x+1)^2}.$

(6) $(xe^x)' = e^x + xe^x = (x+1)e^x.$

(7) $(x^2 \log x)' = 2x \log x + x^2 \cdot \dfrac{1}{x} = x(2\log x + 1).$

(8) $(e^x \cos x)' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$

(9) $(x \tan x)' = \tan x + \dfrac{x}{\cos^2 x}.$

(10) $\left(\dfrac{x}{\sin x}\right)' = \dfrac{\sin x - x\cos x}{\sin^2 x}.$

(11) $\left(\dfrac{1}{\log x}\right)' = -\dfrac{\frac{1}{x}}{(\log x)^2} = -\dfrac{1}{x(\log x)^2}.$

(12) $\left(\dfrac{e^x}{\tan x}\right)' = \dfrac{e^x \tan x - e^x \frac{1}{\cos^2 x}}{\tan^2 x} = \dfrac{e^x(\sin x \cos x - 1)}{\sin^2 x}.$

問 3.

(1) $\{(2x+3)^5\}' = 5(2x+3)^4 \cdot (2x+3)' = 10(2x+3)^4.$

(2) $\{(3x^2-2)^4\}' = 4(3x^2-2)^3 \cdot (3x^2-2)' = 24x(3x^2-2)^3.$

(3) $\left\{\dfrac{(2x+1)^3}{(2x-1)^6}\right\}' = \dfrac{3(2x+1)^2\cdot(2x+1)'(2x-1)^6-(2x+1)^3\cdot 6(2x-1)^5\cdot(2x-1)'}{(2x-1)^{12}}$

$= -\dfrac{6(2x+3)(2x+1)^2}{(2x-1)^7}.$

(4) $(e^{2x})' = e^{2x}\cdot(2x)' = 2e^{2x}.$

(5) $(e^{\frac{1}{x}})' = e^{\frac{1}{x}}\left(\dfrac{1}{x}\right)' = -\dfrac{e^{\frac{1}{x}}}{x^2}.$

(6) $\{\log(1-3x)\}' = \dfrac{(1-3x)'}{1-3x} = -\dfrac{3}{1-3x}.$

(7) $\{\log(x^2+1)\}' = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}.$

(8) $(\sin 3x)' = \cos 3x\cdot(3x)' = 3\cos 3x.$

(9) $(\cos x^2) = -\sin x^2\cdot(x^2)' = -2x\sin x^2.$

(10) $(\tan 4x)' = \dfrac{(4x)'}{\cos^2 4x} = \dfrac{4}{\cos^2 4x}.$

(11) $(\tan\sqrt{x})' = \dfrac{(\sqrt{x}\,)'}{\cos^2\sqrt{x}} = \dfrac{1}{2\sqrt{x}\,\cos^2\sqrt{x}}.$

(12) $(\sin x^2)' = \cos x^2\cdot(x^2)' = 2x\cos x^2.$

(13) $\left(\dfrac{1}{\sin\sqrt{x}}\right)' = -\dfrac{(\sin\sqrt{x}\,)'}{\sin^2\sqrt{x}} = -\dfrac{\cos\sqrt{x}\cdot(\sqrt{x}\,)'}{\sin^2\sqrt{x}} = -\dfrac{\cos\sqrt{x}}{2\sqrt{x}\,\sin^2\sqrt{x}}.$

(14) $\left\{\dfrac{1}{\cos(2x+1)}\right\}' = -\dfrac{\{\cos(2x+1)\}'}{\cos^2(2x+1)} = \dfrac{\sin(2x+1)\cdot(2x+1)'}{\cos^2(2x+1)} = \dfrac{2\sin(2x+1)}{\cos^2(2x+1)}.$

(15) $\left\{\dfrac{1}{\tan(1-x)}\right\}' = -\dfrac{\{\tan(1-x)\}'}{\tan^2(1-x)} = -\dfrac{\frac{(1-x)'}{\cos^2(1-x)}}{\tan^2(1-x)} = \dfrac{1}{\sin^2(1-x)}.$

問 4.

(1) $(\sinh x)' = \dfrac{1}{2}(e^x - e^{-x})' = \dfrac{1}{2}(e^x + e^{-x}) = \cosh x.$

(2) $(\cosh x)' = \dfrac{1}{2}(e^x + e^{-x})' = \dfrac{1}{2}(e^x - e^{-x}) = \sinh x.$

(3) $(\tanh x)' = \left(\dfrac{\sinh x}{\cosh x}\right)' = \dfrac{(\sinh x)'\cosh x - \sinh x(\cosh x)'}{\cosh^2 x}$

$= \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}.$

問 5.

(1) $\left(\sin^{-1}\dfrac{x}{a}\right)' = \dfrac{\left(\frac{x}{a}\right)'}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \dfrac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \dfrac{1}{\sqrt{a^2-x^2}}.$

(2) $\left(\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}\right)' = \dfrac{1}{a}\left(\tan^{-1}\dfrac{x}{a}\right)' = \dfrac{1}{a}\cdot\dfrac{\left(\frac{x}{a}\right)'}{\left(\frac{x}{a}\right)^2+1} = \dfrac{1}{a^2}\cdot\dfrac{1}{\left(\frac{x}{a}\right)^2+1} = \dfrac{1}{x^2+a^2}.$

問 6.

(1) $(\log|\,\log x\,|)' = \dfrac{(\log x)'}{\log x} = \dfrac{1}{x\log x}.$

(2) $(\log|\tan x|)' = \dfrac{(\tan x)'}{\tan x} = \dfrac{1}{\cos^2 x \tan x} = \dfrac{1}{\sin x \cos x}$.

(3) $\left(\log\left|x+\sqrt{x^2+a}\right|\right)' = \dfrac{\left(x+\sqrt{x^2+a}\right)'}{x+\sqrt{x^2+a}} = \dfrac{1+\frac{x}{\sqrt{x^2+a}}}{x+\sqrt{x^2+a}} = \dfrac{1}{\sqrt{x^2+a}}$.

(4) $\left\{(2x+1)^{\frac{1}{3}}\right\}' = \dfrac{1}{3}(2x+1)'(2x+1)^{\frac{1}{3}-1} = \dfrac{2}{3}(2x+1)^{-\frac{2}{3}}$.

(5) $(2^{2x-1})' = 2^{2x-1}\log 2\cdot(2x-1)' = 2\cdot 2^{2x-1}\log 2 = 2^{2x}\log 2$.

(6) $(3^{x-x^2})' = 3^{x-x^2}\log 3\cdot(x-x^2)' = (1-2x)3^{x-x^2}\log 3$.

(7) $y = x^x\ (x>0)$ とおき，対数をとると $\log y = \log x^x = x\log x$ である．両辺を x で微分すると

$$\frac{y'}{y} = \log x + x\cdot\frac{1}{x} = \log x + 1 \qquad \therefore\quad y' = x^x(\log x + 1).$$

(8) $\left(\sin^{-1}\dfrac{x}{2}\right)' = \dfrac{\left(\frac{x}{2}\right)'}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \dfrac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} = \dfrac{1}{\sqrt{4-x^2}}$.

(9) $\left(\sin^{-1}\sqrt{1-x}\right)' = \dfrac{(\sqrt{1-x})'}{\sqrt{1-(1-x)}} = \dfrac{1}{\sqrt{x}}\cdot\dfrac{(1-x)'}{2\sqrt{1-x}} = -\dfrac{1}{2\sqrt{x(1-x)}}$.

(10) $(\cos^{-1}x^2)' = -\dfrac{(x^2)'}{\sqrt{1-(x^2)^2}} = -\dfrac{2x}{\sqrt{1-x^4}}$.

(11) $\left(\tan^{-1}\sqrt{x}\right)' = \dfrac{(\sqrt{x})'}{(\sqrt{x})^2+1} = \dfrac{1}{2\sqrt{x}(x+1)}$.

(12) $\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\dfrac{x}{a}\right)' = \sqrt{a^2-x^2} + x\cdot\dfrac{(a^2-x^2)'}{2\sqrt{a^2-x^2}} + \dfrac{a^2}{\sqrt{a^2-x^2}}$

$= \sqrt{a^2-x^2} - \dfrac{x^2}{\sqrt{a^2-x^2}} + \dfrac{a^2}{\sqrt{a^2-x^2}} = \sqrt{a^2-x^2} + \dfrac{a^2-x^2}{\sqrt{a^2-x^2}} = 2\sqrt{a^2-x^2}$.

問 7.

(1) $\dfrac{dx}{dt} = 2t$，$\dfrac{dy}{dt} = 1 - \dfrac{1}{t^2} = \dfrac{t^2-1}{t^2}$ なので $\dfrac{dy}{dx} = \dfrac{t^2-1}{2t^3}$.

(2) $\dfrac{dx}{dt} = -\sin t + \cos t$，$\dfrac{dy}{dt} = -\sin t - \cos t$ なので $\dfrac{dy}{dx} = \dfrac{\sin t + \cos t}{\sin t - \cos t}$.

問 8.

(1) $y' = \cos x$ なので，点 $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$ における接線の方程式は $y - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right)$ である．また，法線の方程式は $y - \dfrac{1}{2} = -\dfrac{2}{\sqrt{3}}\left(x - \dfrac{\pi}{6}\right)$ である．

(2) $\dot{x} = \dfrac{dx}{dt} = a(1-\cos t)$，$\dot{y} = \dfrac{dy}{dt} = a\sin t$ なので，$t = \dfrac{\pi}{3}$ のとき $x\left(\dfrac{\pi}{3}\right) = a\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}\right)$，$y\left(\dfrac{\pi}{3}\right) = a\left(1 - \dfrac{1}{2}\right) = \dfrac{a}{2}$，$\dot{x}\left(\dfrac{\pi}{3}\right) = \dfrac{a}{2}$，$\dot{y}\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}\,a}{2}$．これより接線の方程式は $y = \sqrt{3}\,x + a\left(2 - \dfrac{\pi}{\sqrt{3}}\right)$ である．また，法線の方程式は $y = -\dfrac{x}{\sqrt{3}} + \dfrac{\pi\, a}{3\sqrt{3}}$ である．