1　極限

1.2　連続関数

問 1　(1)　$\displaystyle\lim_{x\to\sqrt{2}}(x^7-x^3)=8\sqrt{2}-2\sqrt{2}=6\sqrt{2}$

(2)　$\displaystyle\lim_{x\to 8}\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x+1}}=\frac{2\sqrt{2}+\sqrt{2}}{3}=\sqrt{2}$

(3)　$\displaystyle\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=3$

(4)　$\displaystyle\lim_{x\to 2}\frac{x^2-4x+4}{x^2-4}=\lim_{x\to 1}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x-2}{x+2}=\frac{0}{4}=0$

(5)　$\displaystyle\lim_{x\to -2}\frac{x^3+8}{x^2-2x-8}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-4)}=\lim_{x\to 2}\frac{x^2-2x+4}{x-4}=\frac{12}{-6}=-2$

(6)　$\displaystyle\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}\lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}=\frac{1}{4}$

(7)　$\displaystyle\lim_{x\to 1+0}\frac{x-1}{\sqrt{x-1}}=\lim_{x\to 1+0}\sqrt{x-1}=0$

(8)
$$\begin{aligned}\lim_{x\to 2-0}\frac{x-2}{\sqrt[3]{x+6}-2}&=\lim_{x\to 2-0}\frac{(x-2)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}{(\sqrt[3]{x+6}-2)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}\\&=\lim_{x\to 2-0}\frac{(x-2)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}{\left(\sqrt[3]{x+6}\right)^3-2^3}\\&=\lim_{x\to 2-0}\frac{(x-2)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}{x-2}\\&=\lim_{x\to 2-0}\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)\\&=\left(\sqrt[3]{8}\right)^2+2\sqrt[3]{8}+4=2^2+2\cdot 2+4=12\end{aligned}$$

(9)　$\displaystyle\lim_{x\to +0}\left\{\frac{1}{x}\left(\frac{1}{2}+\frac{1}{x-2}\right)\right\}=\lim_{x\to +0}\left\{\frac{1}{x}\cdot\frac{x}{2(x-2)}\right\}=\lim_{x\to +0}\frac{1}{2(x-2)}=-\frac{1}{4}$

問 2　(1)　$\displaystyle\lim_{x\to 2}\frac{1}{(x^2-4)^2}=\infty$

(注意) $\displaystyle\lim_{x\to 2-0}\frac{1}{x^2-4}=\lim_{x\to 2-0}\frac{1}{(x-2)(x+2)}=-\infty$ だから，$\displaystyle\lim_{x\to 2}\frac{1}{x^2-4}\neq\infty$ であることに注意.

(2)　$\displaystyle\lim_{x\to 3-0}\frac{2x}{\sqrt{3-x}}=\lim_{x\to 3-0}\frac{2}{\sqrt{\frac{3}{x}-1}}=\infty$

(3)　$\displaystyle\lim_{x\to\infty}\frac{x-1}{x^2-x}=\lim_{x\to\infty}\frac{x-1}{x(x-1)}=\lim_{x\to\infty}\frac{1}{x}=0$

(4)
$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{x^2+9}-3}&=\lim_{x\to\infty}\frac{x(\sqrt{x^2+9}+3)}{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}=\lim_{x\to\infty}\frac{x(\sqrt{x^2+9}+3)}{x^2+9-9}\\&=\lim_{x\to\infty}\frac{\sqrt{x^2+9}+3}{x}=\lim_{x\to\infty}\left(\sqrt{1+\frac{9}{x^2}}+\frac{3}{x}\right)=1.\end{aligned}$$

(別解) $0<x\to\infty$ に注意して，$\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+9}-3}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{9}{x^2}}-\frac{3}{x}}=\frac{1}{\sqrt{1}-0}=1.$

(5)　$y=-x$ とおくと，$x\to-\infty$ のとき $y\to\infty$．よって，

$$\lim_{x\to-\infty}\frac{x}{|x|}=\lim_{y\to\infty}\frac{-y}{|y|}=\lim_{y\to\infty}\frac{-y}{y}=\lim_{y\to-\infty}-1=-1.$$

(6)　$x^3+2x^2-x-2=(x-1)(x+1)(x+2)\leqq(x+2)^3\to-\infty\quad(x\to-\infty)$ より，

$$\lim_{x\to-\infty}(x^3+2x^2-x-2)=-\infty.$$

(別解 1) $\displaystyle\lim_{x\to-\infty}(x^3+2x^2-x-2)=\lim_{x\to-\infty}(x-1)(x+1)(x+2)=-\infty.$

(別解 2) $y=-x$ とおくと，$x\to-\infty$ のとき $y\to\infty$．よって，

$$\lim_{x\to-\infty}(x^3+2x^2-x-2)=\lim_{y\to\infty}(-y^3+2y^2+y-2)=\lim_{y\to\infty}-(y-1)(y-2)(y+1)=-\infty.$$

(7)　$y=-x$ とおくと，$x\to-\infty$ のとき $y\to\infty$．よって，

$$\lim_{x\to-\infty}\frac{(x-3)^2}{x^3}=\lim_{y\to\infty}\frac{(-y-3)^2}{-y^3}=\lim_{y\to\infty}\frac{(y+3)^2}{-y^3}=\lim_{y\to\infty}-\frac{1}{y}\cdot\left(1+\frac{3}{y}\right)^2=0\cdot1^2=0.$$

(8)　$y=-x$ とおくと，$x\to-\infty$ のとき $y\to\infty$．よって，

$$\begin{aligned}\lim_{x\to-\infty}x(x+\sqrt{x^2-3})&=\lim_{y\to\infty}-y(-y+\sqrt{y^2-3})=\lim_{y\to\infty}y(y-\sqrt{y^2-3})\\&=\lim_{y\to\infty}\frac{y(y-\sqrt{y^2-3})(y+\sqrt{y^2+3})}{y+\sqrt{y^2-3}}\\&=\lim_{y\to\infty}\frac{y\left\{y^2-(y^2-3)\right\}}{y+\sqrt{y^2-3}}=\lim_{y\to\infty}\frac{3y}{y+\sqrt{y^2-3}}\\&=\lim_{y\to\infty}\frac{3}{1+\sqrt{1-\frac{3}{y^2}}}=\frac{3}{1+\sqrt{1}}=\frac{3}{2}.\end{aligned}$$

問 3　(1)　$\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{5x}\right)^x=\lim_{x\to\infty}\left\{\left(1+\frac{1}{5x}\right)^{5x}\right\}^{\frac{1}{5}}=e^{\frac{1}{5}}$

(2)　$\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=\lim_{x\to\infty}\left\{\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right\}^2=e^2$

(3)　$\displaystyle\lim_{x\to\infty}\left(\frac{1}{3}+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(\frac{1}{3}\right)^x\left(1+\frac{3}{x}\right)^x=\lim_{x\to\infty}\left(\frac{1}{3}\right)^x\left\{\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}\right\}^3=0\cdot e^3=0$

(4)　$y=\sqrt{x}$ とおくと，$x\to\infty$ のとき $y\to\infty$．よって，

$$\lim_{x\to\infty}\left(1+\frac{1}{\sqrt{x}}\right)^x=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y^2}=\lim_{y\to\infty}\left\{\left(1+\frac{1}{y}\right)^y\right\}^y.$$

十分大きな y に対して $2<\left(1+\dfrac{1}{y}\right)^y\ (<e)$ だから，$\left\{\left(1+\dfrac{1}{y}\right)^y\right\}^y\geqq2^y\to\infty\quad(y\to\infty)$．よって，

$$\lim_{x\to\infty}\left(1+\frac{1}{\sqrt{x}}\right)^x=\infty.$$

(5)　$\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=\lim_{x\to\infty}\left\{\left(1+\frac{1}{x^2}\right)^{x^2}\right\}^{\frac{1}{x}}$．ここで，$1\leqq\left(1+\dfrac{1}{x^2}\right)^{x^2}\leqq e$ より，

$$1=1^{\frac{1}{x}}\leqq\left\{\left(1+\frac{1}{x^2}\right)^{x^2}\right\}^{\frac{1}{x}}\leqq e^{\frac{1}{x}}\to1\quad(x\to\infty).$$

よって，はさみうちの定理から，$\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = 1$.

(別解 1) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left\{\left(1+\frac{1}{x^2}\right)^{x^2}\right\}^{\frac{1}{x}} = e^0 = 1$

(別解 2) $\displaystyle\lim_{x\to\infty}\left\{\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)^{x^2}\right\} = 0\cdot\log e = 0\cdot 1 = 0$ だから，

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left\{\left(1+\frac{1}{x^2}\right)^{x^2}\right\}^{\frac{1}{x}} = \lim_{x\to\infty}\exp\left\{\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)^{x^2}\right\}$$

$$= \lim_{x\to\infty}\exp\left[\log\left\{\left(1+\frac{1}{x^2}\right)^{x^2}\right\}^{\frac{1}{x}}\right] = e^0 = 0.$$

(6)　$y=-x$ とおくと，$x\to-\infty$ のとき $y\to\infty$．よって，

$$\lim_{x\to-\infty}\left(1-\frac{1}{x}\right)^{3x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{-3y} = \lim_{y\to\infty}\left\{\left(1+\frac{1}{y}\right)^y\right\}^{-3} = e^{-3}.$$

(7)　$\displaystyle\lim_{x\to 0}(1+2x)^{\frac{2}{x}} = \lim_{x\to 0}\left\{(1+2x)^{\frac{1}{2x}}\right\}^4 = e^4$

(8)　$\displaystyle\lim_{x\to 0}(1-x)^{\frac{1}{x}} = e^{-1}$ を認めれば，$\displaystyle\lim_{x\to 0}(1-3x)^{\frac{1}{x}} = \lim_{x\to\infty}\left\{(1-3x)^{\frac{1}{3x}}\right\}^3 = e^{-3}$.

(注意)　$\displaystyle\lim_{x\to 0}(1-x)^{\frac{1}{x}} = e^{-1}$ を示すには，例 6 (2) の証明のようにして，次を示す必要がある：

$$\lim_{x\to+0}(1-x)^{\frac{1}{x}} = \lim_{x\to-0}(1-x)^{\frac{1}{x}} = e^{-1}.$$

(9)　$\displaystyle\lim_{x\to 0}\left(1-4x^2\right)^{\frac{1}{x}} = \lim_{x\to 0}\{(1-2x)(1+2x)\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{(1-2x)^{\frac{1}{x}}(1+2x)^{\frac{1}{x}}\right\} = e^{-2}\cdot e^2 = 1$

問 4　$a=1$ のときは明らか．$0<a<1$ のときは，$f(x)=x^n-a$ は閉区間 $[0,1]$ で連続で，$f(0)=-a<0$，$f(1)=1-a>0$．よって，中間値の定理より $f(c)=0$ となる $c\in[0,1]$ が存在する．$f(x)$ は $x\geqq 0$ で狭義単調増加なので，そのような c はただ一つである．$a>1$ のときは，$b=\frac{1}{a}$ とおけばよい．

問 5　(1)　$5^{\log_5 26}+e^{\log 10} = 26+10 = 36$

(2)　$\log_2\sqrt{128}-\log_3\sqrt{27} = \dfrac{1}{2}\log_2 128-\dfrac{1}{2}\log_3 27 = \dfrac{1}{2}\left(\log_2 2^7-\log_3 3^3\right) = \dfrac{1}{2}(7-3) = 2$

(3)　$\log_2 3+\log_2 6-\log_2 9 = \log_2\dfrac{3\times 6}{9} = \log_2 2 = 1$

(4)　$\log_3 10-\log_3 5-\log_3 6 = \log_3\dfrac{10}{5\times 6} = \log_3\dfrac{1}{3} = \log_3 3^{-1} = -\log_3 3 = -1$

(5)　$\log_{16}\sqrt{32}+\dfrac{\log_{12}27}{\log_{12}3} = \dfrac{\log_2\sqrt{32}}{\log_2 16}+\log_3 27 = \dfrac{\frac{1}{2}\log_2 2^5}{\log_2 2^4}+\log_3 3^3 = \dfrac{1}{2}\cdot\dfrac{5}{4}+3 = \dfrac{29}{8}$

(6)　$(\log_5 49)(\log_7 5) = \dfrac{\log_7 49}{\log_7 5}\dfrac{\log_7 5}{\log_7 7} = \log_7 49 = \log_7 7^2 = 2.$

問 6　(1)　$\displaystyle\lim_{x\to 0}\frac{2}{x}\log(1+2x) = \lim_{x\to 0}4\left\{\frac{1}{2x}\log(1+2x)\right\} = 4\cdot 1 = 4$

(2)　$\displaystyle\lim_{x\to 0}\frac{\log(1-2x)}{x} = \lim_{x\to 0}\left\{-2\cdot\frac{1}{-2x}\log(1-2x)\right\} = -2\cdot 1 = -2$

(別解) $y=-x$ とおけば，$x\to 0$ のとき $y\to 0$．よって，

$$\lim_{x\to 0}\frac{\log(1-2x)}{x}=\lim_{y\to 0}\frac{\log(1+2y)}{-y}=\lim_{y\to 0}\left\{-2\cdot\frac{1}{2y}\log(1+2y)\right\}=-2\cdot 1=-2.$$

(3) $$\lim_{x\to 0}\frac{\log(1-x^2)}{2x}=\lim_{x\to 0}\frac{1}{2x}\log\{(1-x)(1+x)\}=\lim_{x\to 0}\frac{1}{2}\cdot\frac{1}{x}\{\log(1-x)+\log(1+x)\}$$
$$=\lim_{x\to 0}\frac{1}{2}\left\{\frac{1}{x}\log(1-x)+\frac{1}{x}\log(1+x)\right\}=\frac{1}{2}\cdot(-1+1)=0$$

(4) $$\lim_{x\to 0}\frac{\log(3+x)-\log 3}{x}=\lim_{x\to 0}\frac{1}{x}\log\left(\frac{3+x}{3}\right)=\lim_{x\to 0}\frac{1}{x}\log\left(1+\frac{x}{3}\right)$$
$$=\lim_{x\to 0}\frac{1}{3}\left\{\frac{1}{\frac{x}{3}}\log\left(1+\frac{x}{3}\right)\right\}=\frac{1}{3}\cdot 1=\frac{1}{3}$$

(5) $$\lim_{x\to 0}\frac{\log_2(1+x)}{2x}=\lim_{x\to 0}\left\{\frac{1}{2x}\cdot\frac{\log(1+x)}{\log 2}\right\}=\lim_{x\to 0}\left\{\frac{1}{2\log 2}\cdot\frac{\log(1+x)}{x}\right\}=\frac{1}{2\log 2}$$

(6) $$\lim_{x\to 0}\frac{e^{2x}-1}{3x}=\lim_{x\to 0}\left\{\frac{2}{3}\cdot\frac{e^{2x}-1}{2x}\right\}=\frac{2}{3}\cdot 1=\frac{2}{3}$$

(7) $$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\frac{e^{2x}-1}{xe^x}=\lim_{x\to 0}\left\{\frac{2}{e^x}\cdot\frac{e^{2x}-1}{2x}\right\}=\frac{2}{e^0}\cdot 1=2$$

(8) $$\lim_{x\to 0}\frac{3^x-1}{x}=\lim_{x\to 0}\frac{e^{\log 3^x}-1}{x}=\lim_{x\to 0}\frac{\log 3(e^{x\log 3}-1)}{(\log 3)x}$$
$$=\lim_{x\to 0}\left\{\log 3\cdot\frac{e^{(\log 3)x}-1}{(\log 3)x}\right\}=\log 3\cdot 1=\log 3$$

(9) $$\lim_{x\to 0}\frac{2^x-4^x}{x}=\lim_{x\to 0}\frac{2^x(1-2^x)}{x}=\lim_{x\to 0}\frac{-2^x(2^x-1)}{x}=\lim_{x\to 0}\frac{-2^x(e^{x\log 2}-1)}{x}$$
$$=\lim_{x\to 0}\left\{-2^x\cdot\log 2\cdot\frac{e^{(\log 2)x}-1}{(\log 2)x}\right\}=-2^0\cdot(\log 2)\cdot 1=-\log 2$$

問 7 (1) $$\sin\frac{5}{12}\pi=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

(2) $$\cos\frac{7}{12}\pi=\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{3}-\sin\frac{\pi}{4}\sin\frac{\pi}{3}$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{1}{2}-\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}=\frac{1-\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

(3) $$\tan\frac{11}{12}\pi=\tan\left(\frac{2}{3}\pi+\frac{\pi}{4}\right)=\frac{\tan\frac{2}{3}\pi+\tan\frac{\pi}{4}}{1-\tan\frac{2}{3}\pi\cdot\tan\frac{\pi}{4}}$$
$$=\frac{-\sqrt{3}+1}{1-(-\sqrt{3})}=\frac{(1-\sqrt{3})(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}=\frac{4-2\sqrt{3}}{-2}=\sqrt{3}-2$$

(4) $$\sin^2\frac{\pi}{8}=\sin^2\frac{\left(\frac{\pi}{4}\right)}{2}=\frac{1-\cos\frac{\pi}{4}}{2}=\frac{1-\frac{1}{\sqrt{2}}}{2}=\frac{\sqrt{2}-1}{2\sqrt{2}}=\frac{2-\sqrt{2}}{4}$$

(5) $$\sin\frac{7}{12}\pi+\sin\frac{\pi}{12}=2\sin\left(\frac{\frac{7}{12}\pi+\frac{\pi}{12}}{2}\right)\cos\left(\frac{\frac{7}{12}\pi-\frac{\pi}{12}}{2}\right)=2\sin\frac{\pi}{3}\cos\frac{\pi}{4}$$
$$=2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{6}}{2}$$

(6) $$\sin\frac{\pi}{12}+\cos\frac{\pi}{12}=\sqrt{1^2+1^2}\left(\frac{1}{\sqrt{1^2+1^2}}\sin\frac{\pi}{12}+\frac{1}{\sqrt{1^2+1^2}}\cos\frac{\pi}{12}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{\pi}{12} + \frac{1}{\sqrt{2}}\cos\frac{\pi}{12}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\sin\frac{\pi}{12} + \sin\frac{\pi}{4}\cos\frac{\pi}{12}\right)$$

$$= \sqrt{2}\sin\left(\frac{\pi}{12} + \frac{\pi}{4}\right) = \sqrt{2}\sin\left(\frac{4}{12}\pi\right)$$

$$= \sqrt{2}\sin\frac{\pi}{3} = \sqrt{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

問 8 (1) $\displaystyle \lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{2\sin 2x}{2x} = 2$

(2) $\displaystyle \lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left(\frac{\sin 2x}{\sin 3x}\cdot\frac{3x}{2x}\cdot\frac{2}{3}\right) = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\cdot\frac{1}{\frac{\sin 3x}{3x}}\cdot\frac{2}{3}\right) = 1\cdot\frac{1}{1}\cdot\frac{2}{3} = \frac{2}{3}$

(3) $\displaystyle \lim_{x\to 0}\frac{\tan x}{\sin 3x} = \lim_{x\to 0}\left(\frac{\tan x}{\sin 3x}\cdot\frac{3x}{x}\cdot\frac{1}{3}\right) = \lim_{x\to 0}\left(\frac{\tan x}{x}\cdot\frac{1}{\frac{\sin 3x}{3x}}\cdot\frac{1}{3}\right) = 1\cdot\frac{1}{1}\cdot\frac{1}{3} = \frac{1}{3}$

(4) $\displaystyle \lim_{x\to 0}\frac{1-\sin x}{\cos x} = \frac{1-\sin 0}{\cos 0} = \frac{1-0}{1} = 1$

(5) $\displaystyle \lim_{x\to 0}\frac{1-\cos 2x}{x\sin 2x} = \lim_{x\to 0}\frac{(1-\cos 2x)(1+\cos 2x)}{(x\sin 2x)((1+\cos 2x)} = \lim_{x\to 0}\frac{1-\cos^2 2x}{(x\sin 2x)(1+\cos 2x)}$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{(x\sin 2x)(1-\cos 2x)} = \lim_{x\to 0}\frac{2\sin 2x}{2x(1+\cos 2x)}$$

$$= \frac{2\cdot 1}{1+\cos 0} = \frac{2}{1+1} = 1$$

(6) $\displaystyle \lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0}\frac{\sin x - \sin x\cos x}{x^3\cos x}$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{(1-\cos x)(1+\cos x)}{x^2\cos x(1+\cos x)}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{1-\cos^2 x}{x^2\cos x(1+\cos x)}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{\sin^2 x}{x^2\cos x(1+\cos x)}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{\cos x(1+\cos x)}\right\}$$

$$= 1\cdot 1^2\cdot\frac{1}{\cos 0(1+\cos 0)} = \frac{1}{2}$$

(7) $\displaystyle \lim_{x\to 0}\frac{\tan^2 x}{1-\cos x} = \lim_{x\to 0}\frac{\tan^2 x(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{\tan^2 x(1+\cos x)}{1-\cos^2 x}$

$$= \lim_{x\to 0}\frac{\tan^2 x(1+\cos x)}{\sin^2 x} = \lim_{x\to 0}\frac{1+\cos x}{\cos^2 x}$$

$$= \lim_{x\to 0}\frac{1+\cos 0}{\cos^2 0} = 2$$

(8) $\displaystyle \lim_{x\to 0}\frac{\operatorname{cosec} x - \cot x}{x} = \lim_{x\to 0}\frac{\frac{1}{\sin x} - \frac{1}{\tan x}}{x} = \lim_{x\to 0}\frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$

$$= \lim_{x\to 0}\frac{1-\cos x}{x\sin x} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x\sin x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x\sin x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x\sin x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin x}{x(1+\cos x)} = 1\cdot\frac{1}{1+\cos 0} = \frac{1}{2}$$

(9) $\displaystyle\lim_{x\to 0}\frac{\sec\ 2x-1}{\sin^2 x}=\lim_{x\to 0}\frac{\frac{1}{\cos 2x}-1}{\sin^2 x}=\lim_{x\to 0}\frac{1-\cos 2x}{\cos 2x\sin^2 x}=\lim_{x\to 0}\frac{2\sin^2 x}{\cos 2x\sin^2 x}=\lim_{x\to 0}\frac{2}{\cos 2x}=2$

(注意) 途中で，コサイン関数の 2 倍角の公式（$\cos 2x=1-2\sin^2 x$）を使用した．

問 9 (1) $\displaystyle\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

(2) $\displaystyle\cos^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$

(3) $\displaystyle\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

(4) $\displaystyle\sin^{-1}(\sec\pi)=\sin^{-1}\left(\frac{1}{\cos\pi}\right)=\sin(-1)=-\frac{\pi}{2}$

(5) $\cos^{-1}(\cos 3\pi)=\cos(-1)=\pi$ (注意) $\cos^{-1}(\cos 3\pi)\neq 3\pi$

(6) $\displaystyle\sin(\tan^{-1}\sqrt{3})=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

問 10 $\displaystyle\theta=\sin^{-1}\frac{1}{3}=\tan^{-1}x$ とおくと，$\displaystyle\sin\theta=\frac{1}{3}$ かつ $\displaystyle 0<\theta<\frac{\pi}{2}$．これより，

$$1+\tan^2\theta=\frac{1}{\cos^2\theta}=\frac{1}{1-\sin^2\theta}=\frac{9}{8}.$$

よって，$\displaystyle\tan^2\theta=\frac{1}{8}$．ところで，$\displaystyle 0<\theta<\frac{\pi}{2}$ だから，$\displaystyle\tan\theta=\frac{1}{\sqrt{8}}>0$. ゆえに，$\displaystyle x=\tan\theta=\frac{\sqrt{2}}{4}$.

問 11 (1) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}2x}{x}=\lim_{x\to 0}\frac{2\sin^{-1}2x}{2x}=2\cdot 1=2$

(2) $\displaystyle\lim_{x\to 0}\frac{\cos^{-1}x-\frac{\pi}{2}}{x}=\lim_{x\to 0}\frac{-\sin^{-1}x}{x}=-1$

(注意) $\displaystyle\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$ (例 15) を使った．

(3) $y=\tan^{-1}x$ とおくと，$x\to 0$ のとき $y\to 0$．よって，$\displaystyle\lim_{x\to 0}\frac{x}{\tan^{-1}x}=\lim_{y\to 0}\frac{\tan y}{y}=1$.

(4) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}2x}{\tan^{-1}3x}=\lim_{x\to 0}\left\{\frac{\sin^{-1}2x}{2x}\cdot\frac{3x}{\tan^{-1}3x}\cdot\frac{2}{3}\right\}=1\cdot 1\cdot\frac{2}{3}=\frac{2}{3}$

(5) $\displaystyle\lim_{x\to 0}\frac{\sqrt{3}x}{\sin 2x}=\lim_{x\to 0}\left\{\frac{\sqrt{3}}{2}\cdot\frac{2x}{\sin 2x}\right\}=\frac{\sqrt{3}}{2}\cdot 1=\frac{\sqrt{3}}{2}$．よって，$\displaystyle\lim_{x\to 0}\sin^{-1}\frac{\sqrt{3}x}{\sin 2x}=\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$.

(6) $\displaystyle\lim_{x\to 2\pi}\cos^{-1}(\cos x)=\cos^{-1}(\cos 2\pi)=\cos^{-1}1=0$

(注意) $\cos^{-1}(\cos 2\pi)\neq 2\pi$ である．

(7) $\displaystyle\lim_{x\to 0}\frac{\tan\sqrt{3}x}{x}=\lim_{x\to 0}\frac{\sqrt{3}\tan\sqrt{3}x}{\sqrt{3}x}=\sqrt{3}\cdot 1=\sqrt{3}$ だから，$\displaystyle\lim_{x\to 0}\tan^{-1}\frac{\tan\sqrt{3}x}{x}=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$.

(別解) $\displaystyle y=\frac{\tan\sqrt{3}x}{x}$ とおく．$\displaystyle\lim_{x\to 0}\frac{\tan\sqrt{3}x}{x}=\lim_{x\to 0}\frac{\sqrt{3}\tan\sqrt{3}x}{\sqrt{3}x}=\sqrt{3}\cdot 1=\sqrt{3}$．よって，$x\to 0$ のとき $y\to\sqrt{3}$．したがって，$\displaystyle\lim_{x\to 0}\tan^{-1}\frac{\tan\sqrt{3}x}{x}=\lim_{y\to\sqrt{3}}\tan^{-1}y=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$.

(8)　$y=\dfrac{\tan^{-1}x}{2}$ とおくと，$\displaystyle\lim_{x\to\infty}\tan x=\frac{\pi}{2}$ より，$\displaystyle\lim_{x\to\infty}\frac{\tan x}{2}=\frac{\pi}{4}$．よって，$x\to 0$ のとき $y\to\dfrac{\pi}{4}$．これより，$\displaystyle\lim_{x\to\infty}\cos\frac{\tan^{-1}x}{2}=\lim_{y\to\frac{\pi}{4}}\cos y=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$.

(注意) $x\to\infty$ のように無限大が関係する場合は，定理 13 がそのままでは使えない.

(9)　$\displaystyle\lim_{x\to\frac{1}{2}}\cos^2\left(\sin^{-1}x\right)=\cos^2\left(\sin^{-1}\frac{1}{2}\right)=\cos^2\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$

(別解) $\displaystyle\lim_{x\to\frac{1}{2}}\cos^2\left(\sin^{-1}x\right)=\lim_{x\to\frac{1}{2}}\left\{1-\sin^2\left(\sin^{-1}x\right)\right\}=\lim_{x\to\frac{1}{2}}\left[1-\left\{\sin\left(\sin^{-1}x\right)\right\}^2\right]$

$$=\lim_{x\to\frac{1}{2}}(1-x^2)=1-\left(\frac{1}{2}\right)^2=\frac{3}{4}.$$

(注意) $\sin(\sin^{-1}x)=x$ は常に成り立つ．ただし，x の定義域は $-\dfrac{\pi}{2}\leqq x\leqq\dfrac{\pi}{2}$ に注意.

問 12　(1)　$y=\sinh^{-1}x$ とおくと，$x=\sinh y=\dfrac{e^y-e^{-y}}{2}$．$e^{2y}-2xe^y-1=0$ と $e^y>0$ より，$e^y=x+\sqrt{x^2+1}$.

(2)　$y=\cosh^{-1}x$ の定義域と値域に注意すれば (1) と同様.

(3)　$y=\tanh^{-1}x$ の定義域は $|x|<1$．$x=\tanh y=\dfrac{e^y-e^{-y}}{e^y+e^{-y}}$ より，$e^{2y}=\dfrac{1+x}{1-x}$．$|x|<1$ より $e^y=\sqrt{\dfrac{1+x}{1-x}}$.