解答例.

1.(1) $f(x)$ は奇 より $a_n=0$. また.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,dx=\frac{1}{\pi}\left[-\frac{1}{n}x\cos nx\right]_{-\pi}^{\pi}+\frac{1}{\pi n}\int_{-\pi}^{\pi}\cos nx\,dx$$

$$=-\frac{1}{\pi n}\left(\pi(-1)^n-(-\pi)(-1)^n\right)+\frac{1}{\pi n}\left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi}$$

$$=\frac{2}{n}(-1)^{n+1}$$ である

$$\therefore\ f(x)\sim\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

(2). $\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi}=\frac{2}{3}\pi^2$ であり.

$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\sum_{n=1}^{\infty}\frac{4}{n^2}$ なので パーセバルの等式から

$4\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{2}{3}\pi^2$ $\therefore\ \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{6}\pi^2$ である.

2. 関数 $g(x)=x$ が奇 であることを用いる.

$$a_n=\frac{1}{l}\int_{-l}^{l}(2x-3)\cos\frac{\pi nx}{l}dx$$

$$=\frac{2}{l}\int_{-l}^{l}x\cos\frac{\pi nx}{l}dx-\frac{3}{l}\int_{-l}^{l}\cos\frac{\pi nx}{l}dx$$

$$=-\frac{3}{l}\int_{-l}^{l}\cos\frac{\pi nx}{l}dx$$ ここで

$n=0$ のとき. $a_0 = -\frac{3}{l}\int_{-l}^{l} 1 \cdot dx = -6$.

$n \neq 0$ のとき. $a_n = -\frac{3}{l}\int_{-l}^{l} \cos\frac{\pi n x}{l} dx = -\frac{3}{l}\left[\frac{l}{\pi n}\sin\frac{\pi n x}{l}\right]_{-l}^{l} = 0$.

また. $b_n = \frac{1}{l}\int_{-l}^{l}(2x-3)\sin\frac{\pi n x}{l}dx$

$$= \frac{1}{l}\left[-(2x-3)\cdot\frac{l}{\pi n}\cos\frac{\pi n x}{l}\right]_{-l}^{l} + \frac{2}{\pi n}\int_{-l}^{l}\cos\frac{\pi n x}{l}dx$$

$$= -(2l-3)\cdot\frac{1}{\pi n}\cdot(-1)^n + (-2l-3)\frac{1}{\pi n}(-1)^n$$

$$+ \frac{2}{\pi n}\left[\frac{l}{\pi n}\sin\frac{\pi n x}{l}\right]_{-l}^{l}$$

$= \frac{4l}{\pi n}(-1)^{n+1}$ である

$$\therefore f(x) \sim -3 + \sum_{n=1}^{\infty}\frac{4l}{\pi n}(-1)^{n+1}\sin\frac{\pi n x}{l}$$

3. $\hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-iut}dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}|t|e^{-iut}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} -te^{-iut}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{1} te^{-iut}dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{iu}te^{-iut}\right]_{-1}^{0} + \frac{1}{\sqrt{2\pi}}\frac{i}{u}\int_{-1}^{0} e^{-iut}dt$$

$$+ \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{iu}te^{-iut}\right]_{0}^{1} - \frac{1}{\sqrt{2\pi}}\frac{i}{u}\int_{0}^{1} e^{-iut}dt$$

$$= \frac{1}{\sqrt{2\pi}}\left(-\frac{i}{u}e^{iu}\right) + \frac{1}{\sqrt{2\pi}}\frac{i}{u}\left[-\frac{1}{iu}e^{-iut}\right]_{-1}^{0}$$

$$+\frac{1}{\sqrt{2\pi}}\frac{i}{u}e^{-iu} - \frac{1}{\sqrt{2\pi}}\frac{i}{u}\left[-\frac{1}{iu}e^{-iut}\right]_{0}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left(-\frac{i}{u}e^{iu} + \frac{i}{u}e^{-iu} + \frac{1}{u^2}e^{iu} + \frac{1}{u^2}e^{-iu} - \frac{2}{u^2}\right)$$

4. $\mathbf{a}\times\mathbf{b} = (2\times3-1\times1,\ 1\times2-3\times1,\ 1\times1-2\times2)$

$$= (5, -1, -3)$$

$|\mathbf{a}\ \mathbf{b}\ \mathbf{c}| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 1+12-2-4 = 7$. である

5. (1) $\dot{\mathbf{r}}(t) = (-3\sin t, 3\cos t, 4)$ より

$$|\dot{\mathbf{r}}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5.$$

$$\therefore S(t) = \int_0^t |\dot{\mathbf{r}}(\tau)|d\tau = \int_0^t 5\,d\tau = 5t.$$

(2) (1) より $s = 5t$ なので $\mathbf{r}(s) = \left(3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4}{5}s\right)$

である. フレネ標構を求めると

$\mathbf{e}_1(s) = \mathbf{r}'(s) = \left(-\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\right)$ である

$\mathbb{r}''(s) = \left(-\frac{3}{25}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, 0\right)$ と.

$|\mathbb{r}''(s)| = \frac{3}{25}$ より.

$\mathbb{e}_2(s) = \frac{\mathbb{r}''(s)}{|\mathbb{r}''(s)|} = \left(-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0\right)$ である

$\mathbb{e}_3(s) = \mathbb{e}_1(s) \times \mathbb{e}_2(s) = \left(-\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\right) \times \left(-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0\right)$

$= \left(\frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\sin^2\frac{s}{5} + \frac{3}{5}\cos^2\frac{s}{5}\right)$

$= \left(\frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\right)$

6. $\nabla f = \left(\frac{e^x}{1+y^2+z^2}, -\frac{2ye^x}{(1+y^2+z^2)^2}, -\frac{2ze^x}{(1+y^2+z^2)^2}\right)$ より.

$\nabla f(0.1.1) = \left(\frac{1}{3}, -\frac{2}{9}, -\frac{2}{9}\right)$ である.

7. $\operatorname{div} \mathbb{a} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}0 = 0$ である.

$\operatorname{rot} \mathbb{a} = \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x, \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}0, \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right)$

$= (0, 0, 2)$ である.