解答例

1 (1) $f(x)$ は奇 より. $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos nx\,dx$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{\pi n}\left[\frac{1}{n}\sin nx\right]_0^{\pi}$$

$$= \frac{2}{n}(-1)^{n+1}.$$

$$\therefore f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

(2) $$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2.$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{4}{n^2}$$ より

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2}{3}\pi^2 \qquad \therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2.$$

2. $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \frac{\pi}{2}.$$

$n \neq 0$ のとき.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} x e^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{0} x e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx}\right]_0^{\pi} + \frac{1}{2\pi in}\int_0^{\pi} e^{-inx}dx - \frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx}\right]_{-\pi}^{0}$$

$$- \frac{1}{2\pi in}\int_{-\pi}^{0} e^{-inx}dx$$

$$= -\frac{1}{2in}(-1)^n + \frac{1}{2\pi in}\left[-\frac{1}{in}e^{-inx}\right]_0^{\pi}$$

$$\frac{1}{2in}(-1)^n - \frac{1}{2\pi in}\left[-\frac{1}{in}e^{-inx}\right]_{-\pi}^{0}$$

$$= \frac{1}{2\pi n^2}\left((-1)^n - 1 - 1 + (-1)^n\right) = \frac{(-1)^n - 1}{\pi n^2}$$

$$\therefore f(x) \sim \frac{\pi}{2} + \sum_{n \neq 0} \frac{(-1)^n - 1}{\pi n^2} e^{inx}.$$

3. $$\hat{f}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iut}dt = \frac{1}{\sqrt{2\pi}}\int_{-4}^{4}(t-4)e^{-iut}dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{iu}(t-4)e^{-iut}\right]_{-4}^{4} + \frac{1}{\sqrt{2\pi}\cdot iu}\int_{-4}^{4} e^{-iut}dt$$

$$= \frac{8i}{\sqrt{2\pi}u}e^{4iu} + \frac{1}{\sqrt{2\pi}\cdot iu}\left[-\frac{1}{iu}e^{-iut}\right]_{-4}^{4}$$

$$= \frac{8i}{\sqrt{2\pi}u}e^{4iu} + \frac{1}{\sqrt{2\pi}u^2}\left(e^{-4iu} - e^{4iu}\right)$$

4. $\mathbb{a} \times \mathbb{b} = (2\cdot3-1\cdot1,\ 1\cdot2-3\cdot1,\ 1\cdot1-2\cdot2) = (5,-1,-3)$

$|\mathbb{a}\ \mathbb{b}\ \mathbb{c}| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 1+12-2-4 = 7.$

5. $\dot{\mathbb{r}}(t) = (1,2,0)$ より. $|\dot{\mathbb{r}}(t)| = \sqrt{5}$.

$\therefore S(t) = \int_0^t |\dot{\mathbb{r}}(\tau)|\,d\tau = \int_0^t \sqrt{5}\,d\tau = \sqrt{5}\,t.$

6. $\nabla f = \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} f + \frac{\partial}{\partial z} f = 2x+2y+2z.$

$\therefore \nabla f(1.1.1) = 6.$

7. $\frac{\partial}{\partial u}\mathbb{r} = (\cos v, \sin v, 1)$

$\frac{\partial}{\partial v}\mathbb{r} = (-u\sin v, u\cos v, 0)$

$\therefore \mathbb{r}_u \times \mathbb{r}_v = (-u\cos v, u\sin v, u).$

$\therefore |\mathbb{r}_u \times \mathbb{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}\,u$ ($\because$ $0 \le u \le 1$ より)

$\therefore S = \int_0^1 \int_0^{2\pi} |\mathbb{r}_u \times \mathbb{r}_v|\,dv\,du = \int_0^1 \int_0^{2\pi} \sqrt{2}\,u\,dv\,du$

$= 2\pi \int_0^1 \sqrt{2}\,u\,du = 2\pi \left[\frac{\sqrt{2}}{2}u^2\right]_0^1 = \sqrt{2}\,\pi.$

8. $\mathrm{div}\, a = \nabla \cdot a = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y} x + \frac{\partial}{\partial z} 0 = 0$.

$\mathrm{rot}\, a = \nabla \times a = \left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z} x,\ \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x} 0,\ \frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y) \right)$

$= (0, 0, 2)$