問題4.2.1

No. 1/1

4

(1) $\mathbf{r}(t) = (2\cos t,\ 2\sin t)$ を微分すると

$$\dot{\mathbf{r}}(t) = (-2\sin t,\ 2\cos t)$$

$$\therefore\ |\dot{\mathbf{r}}(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

よって、$\mathbf{r}(0)$から$\mathbf{r}(t)$までの弧長sは

$$s = \int_0^t |\dot{\mathbf{r}}(\tau)|\, d\tau = 2t$$

$$\therefore\ t = \frac{s}{2}$$

$$\therefore\ \mathbf{r}(s) = \left(2\cos\frac{s}{2},\ 2\sin\frac{s}{2}\right)$$

(2) $\mathbf{r}(t) = (t,\ 1+2t)$ を微分すると

$$\dot{\mathbf{r}}(t) = (1,\ 2)$$

$$\therefore\ |\dot{\mathbf{r}}(t)| = \sqrt{1+4} = \sqrt{5}$$

よって、$\mathbf{r}(0)$から$\mathbf{r}(t)$までの弧長sは

$$s = \int_0^t |\dot{\mathbf{r}}(\tau)|\, d\tau = \sqrt{5}\, t$$

$$\therefore\ t = \frac{s}{\sqrt{5}}$$

$$\therefore\ \mathbf{r}(s) = \left(\frac{s}{\sqrt{5}},\ 1 + \frac{2}{\sqrt{5}}s\right)$$

問題4.2.2

2

合成関数の微分より

$$\mathbb{r}'(s) = \frac{d}{ds}\mathbb{r}(t(s)) = \frac{d\mathbb{r}}{dt}\cdot\frac{dt}{ds}$$

ここで，$\dot{\mathbb{r}} = \frac{d\mathbb{r}}{dt}$ であり，逆関数の微分より

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\dot{\mathbb{r}}|}$$

なので

$$\mathbb{e}_1 = \mathbb{r}' = \frac{\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|}$$

$$\therefore\ \mathbb{r}'' = \frac{d}{ds}\left(\frac{\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|}\right) = \frac{\left(\frac{d}{ds}\dot{\mathbb{r}}(t(s))\right)|\dot{\mathbb{r}}| - \dot{\mathbb{r}}\frac{d}{ds}|\dot{\mathbb{r}}(t(s))|}{|\dot{\mathbb{r}}|^2}$$

ここで，

$$\frac{d}{ds}\dot{\mathbb{r}}(t(s)) = \frac{d\dot{\mathbb{r}}}{dt}\cdot\frac{dt}{ds} = \frac{\ddot{\mathbb{r}}}{|\dot{\mathbb{r}}|}$$

$$\therefore\ \frac{d}{ds}|\dot{\mathbb{r}}(t(s))| = \frac{d}{ds}\left(|\dot{\mathbb{r}}(t(s))|^2\right)^{1/2} = \frac{1}{2}\left(|\dot{\mathbb{r}}|^2\right)^{-1/2}\cdot 2\left(\frac{d}{ds}\dot{\mathbb{r}}(t(s))\right)\cdot\dot{\mathbb{r}}$$

$$= \frac{\ddot{\mathbb{r}}\cdot\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|^2}$$

$$\therefore\ \mathbb{r}'' = \frac{\ddot{\mathbb{r}} - \frac{(\ddot{\mathbb{r}}\cdot\dot{\mathbb{r}})}{|\dot{\mathbb{r}}|^2}\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|^2} = \frac{|\dot{\mathbb{r}}|^2\ddot{\mathbb{r}} - (\ddot{\mathbb{r}}\cdot\dot{\mathbb{r}})\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|^4}$$

$$= \frac{\ddot{\mathbb{r}}}{|\dot{\mathbb{r}}|^2} - \frac{(\dot{\mathbb{r}}\cdot\ddot{\mathbb{r}})\dot{\mathbb{r}}}{|\dot{\mathbb{r}}|^4}$$

$$\therefore\ \kappa(t) = |\mathbb{r}''| = \frac{1}{|\dot{\mathbb{r}}|^4}\left(|\dot{\mathbb{r}}|^4|\ddot{\mathbb{r}}|^2 - 2|\dot{\mathbb{r}}|^2(\dot{\mathbb{r}}\cdot\ddot{\mathbb{r}})(\dot{\mathbb{r}}\cdot\ddot{\mathbb{r}}) + (\dot{\mathbb{r}}\cdot\ddot{\mathbb{r}})^2|\dot{\mathbb{r}}|^2\right)^{1/2}$$

$$= \frac{1}{|\dot{\mathbb{r}}|^3}\left(|\dot{\mathbb{r}}|^2|\ddot{\mathbb{r}}|^2 - (\dot{\mathbb{r}}\cdot\ddot{\mathbb{r}})^2\right)^{1/2} = \frac{|\dot{\mathbb{r}}\times\ddot{\mathbb{r}}|}{|\dot{\mathbb{r}}|^3}$$

$$\therefore \boldsymbol{e}_2 = \frac{\boldsymbol{r}''}{\kappa} = \frac{1}{\kappa}\left(\frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2} - \frac{(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^4}\right)$$

$$\therefore \boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2 = \frac{1}{|\dot{\boldsymbol{r}}|\kappa}\dot{\boldsymbol{r}} \times \left(\frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2} - \frac{(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^4}\right)$$

$$= \frac{\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^3\kappa} = \frac{\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}|}$$

$$\frac{d}{dt}|\dot{\boldsymbol{r}}|^4 = \frac{d}{dt}(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})^2 = 2(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\frac{d}{dt}(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}) = 4|\dot{\boldsymbol{r}}|^2(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})$$

また、

$$\boldsymbol{r}''' = \frac{d}{ds}\left(\frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2} - \frac{(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^4}\right)$$

$$= \frac{d}{dt}\left(\frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2}\right)\cdot\frac{dt}{ds} - \frac{d}{dt}\left(\frac{(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^4}\right)\cdot\frac{dt}{ds}$$

$$= \frac{|\dot{\boldsymbol{r}}|^2\dddot{\boldsymbol{r}} - 2(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^4}\cdot\frac{1}{|\dot{\boldsymbol{r}}|} - \frac{|\dot{\boldsymbol{r}}|^4\left\{|\ddot{\boldsymbol{r}}|^2\dot{\boldsymbol{r}} + (\dot{\boldsymbol{r}}\cdot\dddot{\boldsymbol{r}})\dot{\boldsymbol{r}} + (\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\ddot{\boldsymbol{r}}\right\} - 4|\dot{\boldsymbol{r}}|^2(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})^2\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^8}\cdot\frac{1}{|\dot{\boldsymbol{r}}|}$$

$$= \frac{|\dot{\boldsymbol{r}}|^2\dddot{\boldsymbol{r}} - 2(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^5} - \frac{\left\{|\dot{\boldsymbol{r}}|^2\left(|\ddot{\boldsymbol{r}}|^2 + (\dot{\boldsymbol{r}}\cdot\dddot{\boldsymbol{r}})\right) - 4(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})^2\right\}\dot{\boldsymbol{r}} + |\dot{\boldsymbol{r}}|^2(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^7}$$

$$= \left(-\frac{|\ddot{\boldsymbol{r}}|^2 + (\dot{\boldsymbol{r}}\cdot\dddot{\boldsymbol{r}})}{|\dot{\boldsymbol{r}}|^5} + \frac{4(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})^2}{|\dot{\boldsymbol{r}}|^7}\right)\dot{\boldsymbol{r}}$$

$$- \frac{3(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})}{|\dot{\boldsymbol{r}}|^5}\ddot{\boldsymbol{r}} + \frac{\dddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^3}$$

$$\therefore |\boldsymbol{r}'\ \boldsymbol{r}''\ \boldsymbol{r}'''| = (\boldsymbol{r}'\times\boldsymbol{r}'')\cdot\boldsymbol{r}'''$$

$$= \left(\frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|}\times\frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2}\right)\cdot\boldsymbol{r}'''$$

ここで,

$$(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) \cdot \ddot{\boldsymbol{r}} = |\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}| = 0$$

などに注意すると

$$|\boldsymbol{r}'\ \boldsymbol{r}''\ \boldsymbol{r}'''| = \left(\frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|} \times \frac{\ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^2}\right) \cdot \frac{\dddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^3}$$

$$= \frac{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) \cdot \dddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|^6} = \frac{|\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ \dddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^6}$$

$$\therefore\ \tau = \frac{1}{\kappa^2}|\boldsymbol{r}'\ \boldsymbol{r}''\ \boldsymbol{r}'''| = \frac{|\dot{\boldsymbol{r}}|^6}{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|^2} \cdot \frac{|\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ \dddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^6}$$

$$= \frac{|\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ \dddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|^2}$$

(1) $\boldsymbol{r}(t) = (a\cos t, b\sin t, 0)$ を微分すると

$$\dot{\boldsymbol{r}}(t) = (-a\sin t, b\cos t, 0)$$

$$\ddot{\boldsymbol{r}}(t) = (-a\cos t, -b\sin t, 0)$$

$$\dddot{\boldsymbol{r}}(t) = (a\sin t, -b\cos t, 0) = -\dot{\boldsymbol{r}}(t)$$

$$\therefore\ |\dot{\boldsymbol{r}}(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = (a^2 - b^2)\sin t\cos t$$

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = (0, 0, ab) \qquad \therefore\ |\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}| = ab$$

$$|\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ \dddot{\boldsymbol{r}}| = |\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}\ -\dot{\boldsymbol{r}}| = 0$$

以上から, $\boldsymbol{e}_1 = \dfrac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|} = \dfrac{(-a\sin t, b\cos t, 0)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$

$$\kappa = ab\,(a^2\sin^2 t + b^2\cos^2 t)^{-3/2}$$

$$e_2 = \frac{1}{\kappa}\left(\frac{\ddot{r}}{|\dot{r}|^2} - \frac{(\dot{r},\ddot{r})\dot{r}}{|\dot{r}|^4}\right)$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}\left(\frac{(-a\cos t, -b\sin t, 0)}{(a^2\sin^2 t + b^2\cos^2 t)} - \frac{(a^2-b^2)\sin t\cos t(-a\sin t, b\cos t, 0)}{(a^2\sin^2 t + b^2\cos^2 t)^2}\right)$$

$$= \frac{1}{ab\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\Big\{(-a^3\sin^2 t\cos t - ab^2\cos^3 t, -a^2b\sin^3 t - b^3\sin t\cos^2 t, 0)$$

$$- (-a^3\sin^2 t\cos t + ab^2\sin^2 t\cos t, a^2b\sin t\cos^2 t - b^3\sin t\cos^2 t, 0)\Big\}$$

$$= \frac{1}{ab\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-ab^2\cos t(\cos^2 t + \sin^2 t), -a^2b\sin t(\sin^2 t + \cos^2 t), 0)$$

$$= \frac{(-b\cos t, -a\sin t, 0)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

$$e_3 = \frac{\dot{r}\times\ddot{r}}{|\dot{r}\times\ddot{r}|} = \frac{(0,0,ab)}{ab} = (0,0,1)$$

$$\tau = \frac{|\dot{r}\ \ddot{r}\ \dddot{r}|}{|\dot{r}\times\ddot{r}|^2} = 0$$

(2) $r(t) = (\cosh t, \sinh t, t)$ を微分すると、

$$\dot{r}(t) = (\sinh t, \cosh t, 1)$$

$$\ddot{r}(t) = (\cosh t, \sinh t, 0)$$

$$\dddot{r}(t) = (\sinh t, \cosh t, 0)$$

$$|\dot{r}| = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$= \sqrt{1 + \cosh 2t}$$

$$(\cosh t)' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = \sinh t$$

$$(\sinh t)' = \left(\frac{e^t - e^{-t}}{2}\right)' = \frac{e^t + e^{-t}}{2} = \cosh t$$

ここで,

$$\sinh^2 t + \cosh^2 t = \frac{e^{2t} - 2 + e^{-2t}}{4} + \frac{e^{2t} + 2 + e^{-2t}}{4}$$

$$= \frac{e^{2t} + e^{-2t}}{2} = \cosh 2t$$

を用いた.

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \sinh t \cosh t + \cosh t \sinh t = 2 \sinh t \cosh t$$

$$= 2 \left(\frac{e^{t} - e^{-t}}{2} \right) \left(\frac{e^{t} + e^{-t}}{2} \right)$$

$$= \frac{e^{2t} - e^{-2t}}{2} = \sinh 2t$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \left(\begin{vmatrix} \cosh t & 1 \\ \sinh t & 0 \end{vmatrix}, \begin{vmatrix} 1 & \sinh t \\ 0 & \cosh t \end{vmatrix}, \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix} \right)$$

$$= \left(-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t \right)$$

$$= \left(-\sinh t, \cosh t, -1 \right)$$

ここで,

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1$$

となることを用いた.

$$\left| \dot{\mathbf{r}} \; \ddot{\mathbf{r}} \; \dddot{\mathbf{r}} \right| = \left(\dot{\mathbf{r}} \times \ddot{\mathbf{r}} \right) \cdot \dddot{\mathbf{r}} = -\sinh^2 t + \cosh^2 t = 1.$$

以上から, $\mathbf{e}_1 = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \dfrac{(\sinh t, \cosh t, 1)}{\sqrt{1 + \cosh 2t}}$

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{\sinh^2 t + \cosh^2 t + 1}}{(1 + \cosh 2t)^{3/2}} = \frac{1}{1 + \cosh 2t}$$

$$\mathbf{e}_2 = \frac{1}{\kappa}\left(\frac{\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} - \frac{(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^4}\right)$$

$$= (1+\cosh 2t)\left(\frac{(\cosh t, \sinh t, 0)}{1+\cosh 2t} - \frac{\sinh 2t\,(\sinh t, \cosh t, 1)}{(1+\cosh 2t)^2}\right)$$

$$= \frac{1}{1+\cosh 2t}\Big((1+\cosh 2t)(\cosh t, \sinh t, 0) - \sinh 2t\,(\sinh t, \cosh t, 1)\Big)$$

$$= \frac{1}{1+\cosh 2t}\Big(\cosh t + \cosh 2t\cosh t - \sinh 2t\sinh t,\ \sinh t + \cosh 2t\sinh t - \sinh 2t\cosh t,\ -\sinh 2t\Big)$$

ここで

$$\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

$$= \frac{1}{4}\left\{(e^{\alpha}+e^{-\alpha})(e^{\beta}+e^{-\beta}) - (e^{\alpha}-e^{-\alpha})(e^{\beta}-e^{-\beta})\right\}$$

$$= \frac{1}{4}\left(2e^{-\alpha}e^{\beta} + 2e^{\alpha}e^{-\beta}\right) = \frac{e^{\alpha-\beta}+e^{-(\alpha-\beta)}}{2}$$

$$= \cosh(\alpha-\beta)$$

$$\cosh\alpha\sinh\beta - \sinh\alpha\cos\beta$$

$$= \frac{1}{4}\left\{(e^{\alpha}+e^{-\alpha})(e^{\beta}-e^{-\beta}) - (e^{\alpha}-e^{-\alpha})(e^{\beta}+e^{-\beta})\right\}$$

$$= \frac{-e^{\alpha-\beta}+e^{-(\alpha-\beta)}}{2}$$

$$= -\sinh(\alpha-\beta)$$

なので

$$\mathbf{e}_2 = \frac{1}{1+\cosh 2t}\left(2\cosh t,\ 0,\ -\sinh 2t\right)$$

$$\mathbb{e}_3 = \frac{\dot{\mathbb{r}} \times \ddot{\mathbb{r}}}{|\dot{\mathbb{r}} \times \ddot{\mathbb{r}}|} = \frac{(-\sinh t, \cosh t, -1)}{\sqrt{1+\cosh 2t}}$$

$$\tau = \frac{|\dot{\mathbb{r}}\ \ddot{\mathbb{r}}\ \dddot{\mathbb{r}}|}{|\dot{\mathbb{r}} \times \ddot{\mathbb{r}}|^2} = \frac{1}{1+\cosh 2t}$$

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(1) $\mathbf{r}(t) = (2\cos t, 2\sin t)$ を微分すると

$\dot{\mathbf{r}} = (-2\sin t, 2\cos t)$, $\ddot{\mathbf{r}} = (-2\cos t, -2\sin t)$,

$|\dot{\mathbf{r}}| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$, $|\ddot{\mathbf{r}}| = 2$

$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 4\sin t\cos t - 4\cos t\sin t = 0$

$\therefore\ \kappa = \dfrac{\sqrt{4 \cdot 4 - 0^2}}{2^3} = \dfrac{4}{8} = \dfrac{1}{2}$

(2) $\mathbf{r}(t) = (\cosh t, \sinh t)$ を微分すると

$\dot{\mathbf{r}} = (\sinh t, \cosh t)$, $\ddot{\mathbf{r}} = (\cosh t, \sinh t)$,

$|\dot{\mathbf{r}}| = \sqrt{\sinh^2 t + \cosh^2 t} = \sqrt{\cosh 2t}$,

$|\ddot{\mathbf{r}}| = \sqrt{\cosh 2t}$

$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 2\sinh t\cosh t = \sinh 2t$

$\therefore\ \kappa = \dfrac{\sqrt{\cosh^2 2t - \sinh^2 2t}}{(\cosh 2t)^{3/2}} = \dfrac{1}{(\cosh 2t)^{3/2}}$

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$\mathbf{r}(t) = (t, f(t))$ を微分すると

$\dot{\mathbf{r}} = (1, \dot{f})$, $\ddot{\mathbf{r}} = (0, \ddot{f})$

$\therefore$ $|\dot{\mathbf{r}}| = \sqrt{1+(\dot{f})^2}$, $|\ddot{\mathbf{r}}| = |\ddot{f}|$, $\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \dot{f}\ddot{f}$

よって,

$$|\dot{\mathbf{r}}|^2|\ddot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2 = \{1+(\dot{f})^2\}|\ddot{f}|^2 - (\dot{f}\ddot{f})^2 = |\ddot{f}|^2$$

なので, 問題3の公式より

$$\kappa = \frac{\sqrt{|\dot{\mathbf{r}}|^2|\ddot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}}{|\dot{\mathbf{r}}|^3} = \frac{|\ddot{f}|}{\{1+(\dot{f})^2\}^{3/2}}$$

(1) $\mathbf{r}(t) = (t, t^2)$ について, $f(t) = t^2$ とすると

$$\kappa = \frac{2}{(1+4t^2)^{3/2}}$$

(2) $\mathbf{r}(t) = (t, e^{-t})$ について, $f(t) = e^{-t}$ とすると

$$\kappa = \frac{e^{-t}}{(1+e^{-2t})^{3/2}}$$

問題 4.3.1

1.

$r = \sqrt{x^2+y^2+z^2}$ なので

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2} 2x$$

$$= \frac{x}{r}$$

などから

$$\nabla r = \frac{\boldsymbol{r}}{r}$$

を得る.

問1の合成関数の微分 (4.24) を, $g(\boldsymbol{r}) = r$ として応用すると,

$$\nabla f(r) = \frac{df}{dr}\nabla r$$

となることを用いるとよい.

(1) $\nabla r^n = \frac{d}{dr} r^n \nabla r = n r^{n-1} \cdot \frac{\boldsymbol{r}}{r} = n r^{n-2}\boldsymbol{r}$

(2) $\nabla\left(\frac{e^r}{r}\right) = \frac{d}{dr}\left(\frac{e^r}{r}\right)\nabla r = \frac{e^r r - e^r}{r^2}\cdot\frac{\boldsymbol{r}}{r} = \frac{e^r(r-1)}{r^3}\boldsymbol{r}$

(3) $\nabla \log r = \frac{d}{dr}(\log r)\nabla r = \frac{1}{r}\cdot\frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{r}}{r^2}$

問題4.3.1

2

(1) $f(x,y,z)=\dfrac{e^x}{1+y^2+z^2}$ について

$$\nabla f=\left(\frac{e^x}{1+y^2+z^2},\ \frac{-2ye^x}{(1+y^2+z^2)^2},\ \frac{-2ze^x}{(1+y^2+z^2)^2}\right)$$

$$=\frac{e^x}{(1+y^2+z^2)^2}\left(1+y^2+z^2,\ -2y,\ -2z\right)$$

(2) $f(x,y,z)=x^2+y^2+z^2$ について,

$$\nabla f=(2x,\ 2y,\ 2z)=2\boldsymbol{r}$$

(3) $f(x,y,z)=x^2y+xy^2-z$ について

$$\nabla f=(2xy+y^2,\ x^2+2xy,\ -1)$$

(4) $f(x,y,z)=x^2\sin y\cos z$ について

$$\nabla f=(2x\sin y\cos z,\ x^2\cos y\cos z,\ -x^2\sin y\sin z)$$

問題 4.3.2

2

(1) $a = (x^2, y^2, z^2)$ について

$$\mathrm{div}\, a = 2x + 2y + 2z = 2(x+y+z)$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ y^2 & z^2 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ z^2 & x^2 \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ x^2 & y^2 \end{vmatrix} \right)$$

$$= \left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}, \frac{\partial x^2}{\partial z} - \frac{\partial z^2}{\partial x}, \frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y} \right) = 0$$

∴ a は渦なし

(2) $a = (-y, x, 0)$ について

$$\mathrm{div}\, a = 0 + 0 + 0 = 0$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ x & 0 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ 0 & -y \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ -y & x \end{vmatrix} \right) = \left(0, 0, \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) = (0, 0, 2)$$

∴ a は湧き出しなし

(3) $a = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$ について,

$$\mathrm{div}\, a = \frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ \frac{x}{x^2+y^2} & 0 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ 0 & \frac{-y}{x^2+y^2} \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \right)$$

$$= \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \right)$$

$$= \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right) = 0$$

∴ a は湧き出しなし，渦なし

(4) $a = (y^2z, xz^2, x^2y)$ について

$\mathrm{div}\, a = 0 + 0 + 0 = 0$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ xz^2 & x^2y \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ x^2y & y^2z \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ y^2z & xz^2 \end{vmatrix} \right)$$

$$= \left(x^2 - 2xz,\ y^2 - 2xy,\ z^2 - 2yz \right)$$

∴ aは 湧き出しなし

問題 4.3.2

3

(1) $\mathrm{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\cdot\nabla r = -\frac{\boldsymbol{r}}{r^3}$

(2) 例題 4.3.3 (2) を使うとよい.

$$f(\boldsymbol{r}) = \frac{1}{r^3},\quad \boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{r}$$

とすると,

$$\nabla f = -\frac{3}{r^4}\nabla r = -\frac{3\boldsymbol{r}}{r^5},\quad \nabla\cdot\boldsymbol{a} = 3$$

$$\therefore\ \mathrm{div}\left(\frac{\boldsymbol{r}}{r^3}\right) = \nabla\cdot(f\boldsymbol{a}) = (\nabla f)\cdot\boldsymbol{a} + f(\nabla\cdot\boldsymbol{a})$$

$$= -\frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r} + \frac{1}{r^3}\cdot 3$$

$$= -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

(3) $\boldsymbol{\omega}$ が定ベクトルであることに注意して 問2の (2) を使うとよい.

$$\mathrm{rot}(\boldsymbol{\omega}\times\boldsymbol{r}) = \nabla\times(\boldsymbol{\omega}\times\boldsymbol{r})$$

$$= (\boldsymbol{r}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{r} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{r}) - \boldsymbol{r}(\nabla\cdot\boldsymbol{\omega})$$

$$= -(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{r} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{r})$$

$$= -\left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z}\right)\boldsymbol{r} + 3\boldsymbol{\omega}$$

$$= -(\omega_1, 0, 0) - (0, \omega_2, 0) - (0, 0, \omega_3) + 3\boldsymbol{\omega}$$

$$= -\boldsymbol{\omega} + 3\boldsymbol{\omega}$$

$$= 2\boldsymbol{\omega}$$

問題4.4.1

1

(1) $\mathbf{a} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right)$, $\mathbf{r}(t) = (\cos \pi t, \sin \pi t, t)$ について

$$\mathbf{a}(\mathbf{r}(t)) = (-\sin \pi t, \cos \pi t, t)$$

$$\dot{\mathbf{r}}(t) = (-\pi \sin \pi t, \pi \cos \pi t, 1)$$

$$\therefore \int_C \mathbf{a} \cdot d\mathbf{r} = \int_0^1 \mathbf{a}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\, dt$$

$$= \int_0^1 \left\{ \pi (\sin^2 \pi t + \cos^2 \pi t) + t \right\} dt$$

$$= \int_0^1 (\pi + t)\, dt = \left[\pi t + \frac{t^2}{2} \right]_0^1$$

$$= \pi + \frac{1}{2}$$

(2) $\mathbf{a} = (xz^2, yz^2, z)$, $\mathbf{r}(t) = (t^2, t^2, t)$ について

$$\mathbf{a}(\mathbf{r}(t)) = (t^4, t^4, t)$$

$$\dot{\mathbf{r}}(t) = (2t, 2t, 1)$$

$$\therefore \int_C \mathbf{a} \cdot d\mathbf{r} = \int_0^1 \mathbf{a}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\, dt$$

$$= \int_0^1 (2t^5 + 2t^5 + t)\, dt$$

$$= \int_0^1 (4t^5 + t)\, dt = \left[\frac{4t^6}{6} + \frac{t^2}{2} \right]_0^1$$

$$= \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

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