問題 4.3.2

2

(1) $a=(x^2, y^2, z^2)$ について

$$\mathrm{div}\, a = 2x+2y+2z = 2(x+y+z)$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ y^2 & z^2 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ z^2 & x^2 \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ x^2 & y^2 \end{vmatrix} \right)$$

$$= \left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}, \frac{\partial x^2}{\partial z} - \frac{\partial z^2}{\partial x}, \frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y} \right) = 0$$

∴ aは渦なし

(2) $a=(-y, x, 0)$ について

$$\mathrm{div}\, a = 0+0+0=0$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ x & 0 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ 0 & -y \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ -y & x \end{vmatrix} \right) = \left(0, 0, \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) = (0,0,2)$$

∴ aは湧き出しなし

(3) $a = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$ について,

$$\mathrm{div}\, a = \frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ \frac{x}{x^2+y^2} & 0 \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ 0 & \frac{-y}{x^2+y^2} \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \right)$$

$$= \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \right)$$

$$= \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right) = 0$$

∴ aは湧き出しなし，渦なし

(4) $a = (y^2 z, xz^2, x^2 y)$ について

$$\mathrm{div}\, a = 0 + 0 + 0 = 0$$

$$\mathrm{rot}\, a = \left(\begin{vmatrix} \nabla_2 & \nabla_3 \\ xz^2 & x^2 y \end{vmatrix}, \begin{vmatrix} \nabla_3 & \nabla_1 \\ x^2 y & y^2 z \end{vmatrix}, \begin{vmatrix} \nabla_1 & \nabla_2 \\ y^2 z & xz^2 \end{vmatrix} \right)$$

$$= \left(x^2 - 2xz,\ y^2 - 2xy,\ z^2 - 2yz \right)$$

∴ aは湧き出しなし

問題 4.3.2

3

(1) $\operatorname{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\cdot\nabla r = -\frac{\boldsymbol{r}}{r^3}$

(2) 例題 4.3.3 (2) を使うとよい.

$$f(\boldsymbol{r}) = \frac{1}{r^3}, \quad \boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{r}$$

とすると,

$$\nabla f = -\frac{3}{r^4}\nabla r = -\frac{3\boldsymbol{r}}{r^5}, \quad \nabla\cdot\boldsymbol{a} = 3$$

$$\therefore \operatorname{div}\left(\frac{\boldsymbol{r}}{r^3}\right) = \nabla\cdot(f\boldsymbol{a}) = (\nabla f)\cdot\boldsymbol{a} + f(\nabla\cdot\boldsymbol{a})$$

$$= -\frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r} + \frac{1}{r^3}\cdot 3$$

$$= -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

(3) $\boldsymbol{\omega}$ が定ベクトルであることに注意して 問2の (2) を使うとよい.

$$\operatorname{rot}(\boldsymbol{\omega}\times\boldsymbol{r}) = \nabla\times(\boldsymbol{\omega}\times\boldsymbol{r})$$

$$= (\boldsymbol{r}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{r} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{r}) - \boldsymbol{r}(\nabla\cdot\boldsymbol{\omega})$$

$$= -(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{r} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{r})$$

$$= -\left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z}\right)\boldsymbol{r} + 3\boldsymbol{\omega}$$

$$= -(\omega_1, 0, 0) - (0, \omega_2, 0) - (0, 0, \omega_3) + 3\boldsymbol{\omega}$$

$$= -\boldsymbol{\omega} + 3\boldsymbol{\omega}$$

$$= 2\boldsymbol{\omega}$$