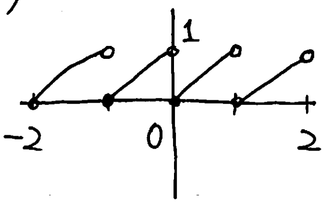
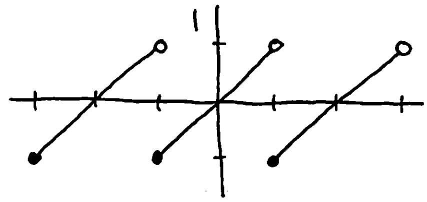
3.1節

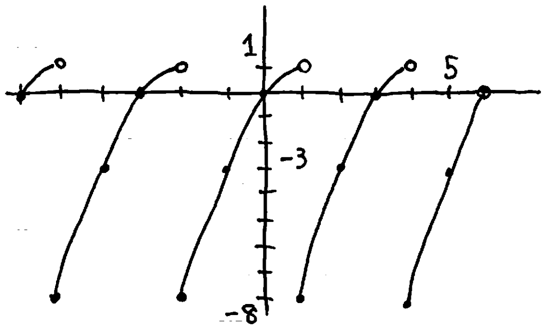
1.(1)



(2)



(3) $f(x) = -x^2 + 2x = -(x-1)^2 + 1$ より

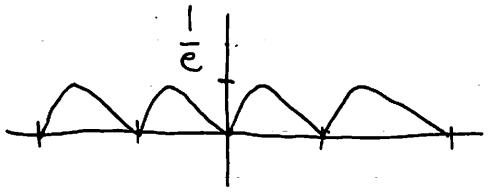


となる

(4). $\lim_{x \to +0} -x \log x = 0$ であり.また. $f'(x) = -\log x - 1$ より

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$	1
$f(x)$	0に近づく	↗	$\frac{1}{e}$	↘	0
$f'(x)$		+	0	−	

となることから.



となる.

2(1). $f(x)$は(原点を除いて)奇関数より

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0.$$ である.また.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} \sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{-2}{\pi n}((-1)^n - 1) = \frac{2}{\pi n}(1 - (-1)^n)$$ より

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi n}(1-(-1)^n)\sin nx$$ となる.

(2) $f(x)$は奇関数より. $a_n=0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{2}{\pi n}\left[x\cos nx\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos nx\,dx$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{\pi n^2}\left[\sin nx\right]_0^{\pi} = \frac{2}{n}(-1)^{n+1}.$$

$$\therefore f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

(3) a_n, b_nを計算すると.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3)\,dx = \frac{1}{\pi}\left[x^2-3x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\pi^2-3\pi-\pi^2-3\pi\right) = -6$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3)\cos nx\,dx = \frac{1}{\pi}\left[\frac{1}{n}(2x-3)\sin nx\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}\sin nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3)\sin nx\,dx = \frac{1}{\pi}\left[-\frac{1}{n}(2x-3)\cos nx\right]_{-\pi}^{\pi} + \frac{2}{\pi n}\int_{-\pi}^{\pi}\cos nx\,dx$$

$$= -\frac{1}{n\pi}(-1)^n\cdot\left((2\pi-3)-(-2\pi-3)\right) + \frac{2}{\pi n^2}\left[\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{4}{n}(-1)^{n+1}$$ となり

$$f(x) \sim -3 + \sum_{n=1}^{\infty}\frac{4}{n}(-1)^{n+1}\sin nx$$ となる.

(4) a_n, b_nを計算すると.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2-3x\,dx = \frac{1}{\pi}\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{1}{3}\pi^3-\frac{3}{2}\pi^2+\frac{1}{3}\pi^3+\frac{3}{2}\pi^2\right)$$

$$= \frac{2}{3}\pi^2$$

$x\cos nx$ が奇関数なので.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2-3x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x^2\sin nx\right]_0^{\pi} - \frac{4}{\pi n}\int_0^{\pi} x\sin nx\,dx$$

$$= -\frac{4}{\pi n}\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{4}{n^2}(-1)^n - \frac{4}{\pi n^2}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = \frac{4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2-3x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}-3x\sin nx\,dx = \frac{-6}{\pi}\int_0^{\pi}x\sin nx\,dx$$

$= \frac{6}{n}(-1)^n$ ← (2)の計算と同じなので省略.

$\therefore f(x) \sim \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \frac{6}{n}(-1)^n\sin nx$ となる.

(5) $\sin^2 x = \frac{1-\cos 2x}{2}$ より $f(x) = \frac{1}{2} - \frac{\cos 2x}{2}$ である.

(6) $A = \int_{-\pi}^{\pi}e^x\cos nx\,dx$ とすると. $(n \neq 0)$

$$A = \left[\frac{1}{n}e^x\sin nx\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}e^x\sin nx\,dx = -\frac{1}{n}\int_{-\pi}^{\pi}e^x\sin nx\,dx$$

$$= -\frac{1}{n}\left[-\frac{1}{n}e^x\cos nx\right]_{-\pi}^{\pi} - \frac{1}{n^2}\int_{-\pi}^{\pi}e^x\cos nx\,dx$$

$$= \frac{1}{n^2}(-1)^n(e^{\pi}-e^{-\pi}) - \frac{1}{n^2}A$$

$\therefore A = \frac{1}{n^2+1}(-1)^n(e^{\pi}-e^{-\pi})$ また.この計算の一行目から

$\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\sin nx\,dx = -\frac{n}{\pi}A = \frac{n}{(n^2+1)\pi}(-1)^{n+1}(e^{\pi}-e^{-\pi})$ となる.さらに.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi}-e^{-\pi})$ より

$$f(x) \sim \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx + \frac{n}{n^2+1}(-1)^{n+1}\sin nx\right\}$$

となる.

No. · · ( )

3(1) $f(x)$ のフーリエ級数を求めると．偶関数より $b_n=0$．

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left(\frac{1}{3}\pi^3+\frac{1}{3}\pi^3\right)=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx=\frac{4}{n^2}(-1)^n$$ ← 2(4)の計算から．

$$\therefore f(x)\sim\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$ となる．

$f(x)$ は $x=0$ で連続なので．x に 0 を代入すると．

$$0=f(0)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$ となり．

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{1}{12}\pi^2$$ となる．

(2). 2(2)より $f(x)\sim\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$ である．

$f(x)$ は $x=\frac{\pi}{2}$ で連続なので．x に $\frac{\pi}{2}$ を代入すると．

$$\frac{\pi}{2}=f\left(\frac{\pi}{2}\right)=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin\frac{\pi n}{2}=\sum_{n=1}^{\infty}\frac{2}{2n-1}(-1)^{(2n-1)+1}\sin\frac{\pi(2n-1)}{2}$$

n が偶数だと．$\sin\frac{\pi n}{2}=0$ より奇数のところだけぬき出している

$$=\sum_{n=1}^{\infty}\frac{2}{2n-1}\sin\left(\pi n-\frac{\pi}{2}\right)=\sum_{n=1}^{\infty}\frac{2}{2n-1}(-1)^{n+1}$$ となる．

$\cos n\pi=(-1)^n$ と似た考察をする．

$$\therefore \sum_{n=1}^{\infty}\frac{1}{2n-1}(-1)^{n+1}=\frac{\pi}{4}$$ である．

3.2 節

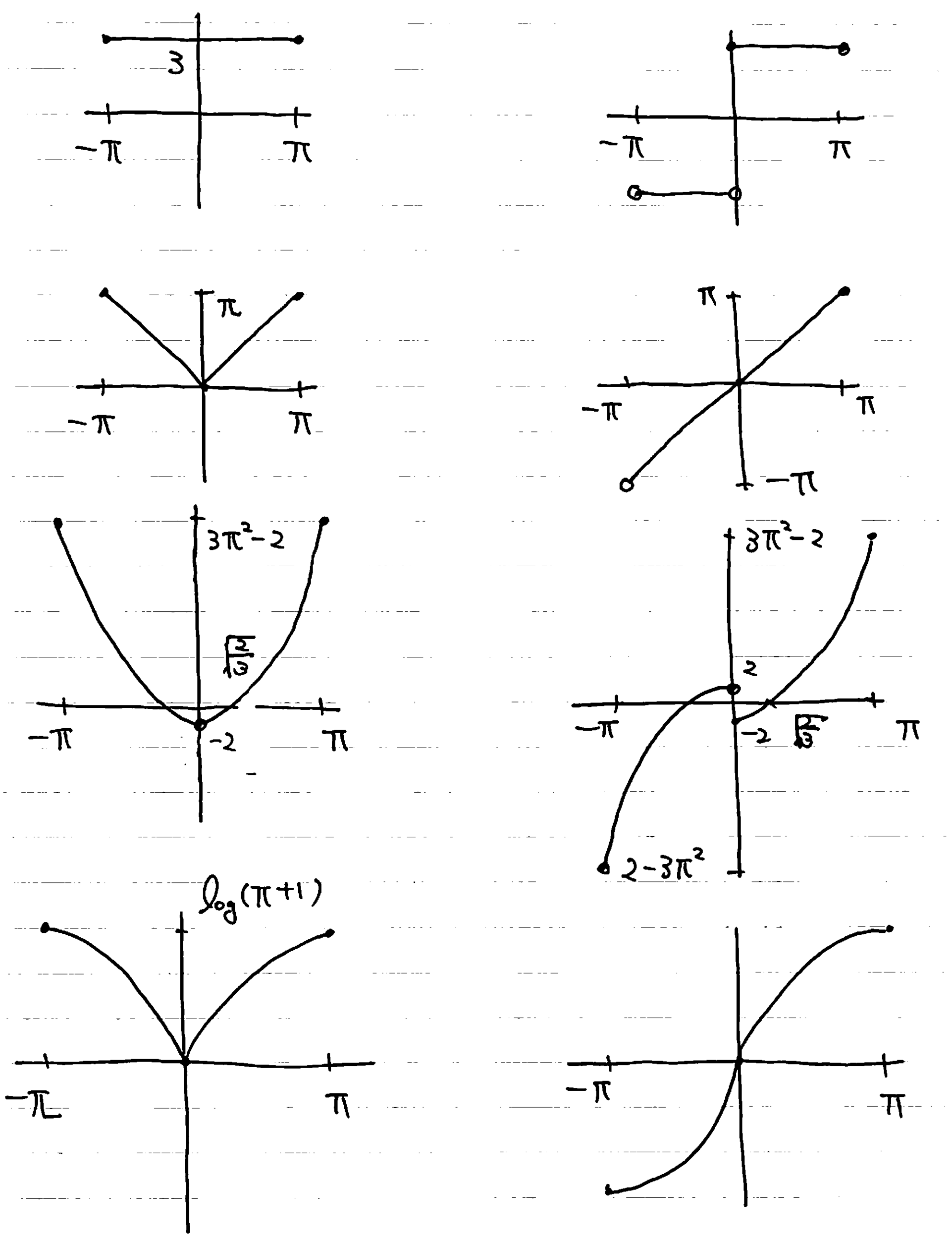
1. $f(-x)=f(x)$ $\qquad\qquad$ $f(-x)=-f(x)$

(1)

(2)

(3)

(4)



2(1) 余弦 $\quad a_0=\frac{2}{\pi}\int_0^\pi \pi-x\,dx=\frac{2}{\pi}\left[\pi x-\frac{1}{2}x^2\right]_0^\pi=\frac{2}{\pi}\left(\pi^2-\frac{1}{2}\pi^2\right)=\pi.$

$$a_n=\frac{2}{\pi}\int_0^\pi(\pi-x)\cos nx\,dx=\frac{2}{\pi}\left[\frac{1}{n}(\pi-x)\sin nx\right]_0^\pi+\frac{2}{\pi n}\int_0^\pi \sin nx\,dx$$

$$=\frac{2}{\pi n}\left[-\frac{1}{n}\cos nx\right]_0^\pi=-\frac{2}{\pi n^2}\left((-1)^n-1\right)=\frac{2}{\pi n^2}\left(1-(-1)^n\right)$$

$$\therefore\ f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1-(-1)^n\right)\cos nx$$

正弦

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}(\pi-x)\cos nx\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{2}{n} - \frac{2}{\pi n}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = \frac{2}{n}$$ よ')

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n}\sin nx$$

(2) 余弦 $a_0 = \frac{2}{\pi}\int_0^{\pi} 1\,dx = 2$, $a_n = \frac{2}{\pi}\int_0^{\pi}\cos nx\,dx = \left[\frac{2}{\pi n}\sin nx\right]_0^{\pi} = 0$

$$\therefore\ f(x) \sim 1$$

正弦

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{\pi n}\left(1-(-1)^n\right)$$

$$\therefore\ f(x) \sim \sum_{n=1}^{\infty}\frac{2}{\pi n}\left(1-(-1)^n\right)\sin nx$$

(3) 余弦 $a_0 = \frac{2}{\pi}\int_0^{\pi} -x-2\,dx = \frac{2}{\pi}\left[-\frac{1}{2}x^2 - 2x\right]_0^{\pi} = \frac{2}{\pi}\left(-\frac{1}{2}\pi^2 - 2\pi\right)$

$$= -\pi - 4$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(-x-2)\cos nx\,dx = \frac{2}{\pi}\left[\frac{1}{n}(-x-2)\sin nx\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{2}{\pi n}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{1}{\pi n^2}\left(1-(-1)^n\right)$$

$$\therefore\ f(x) \sim -\frac{1}{2}\pi - 2 + \sum_{n=1}^{\infty}\frac{1}{\pi n^2}\left(1-(-1)^n\right)\cos nx$$

正弦 $b_n = \frac{2}{\pi}\int_0^{\pi}(-x-2)\sin nx\,dx = \frac{2}{\pi}\left[\frac{1}{n}(x+2)\cos nx\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}\cos nx\,dx$

No.

$= \frac{2}{\pi n}\left((\pi+2)(-1)^n - 2\right) - \frac{2}{\pi n}\left[\frac{1}{n}\sin nx\right]_0^\pi$ より

$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi n}\left((\pi+2)(-1)^n-2\right)\sin nx$. である.

(4) 余弦 $a_0 = \frac{2}{\pi}\int_0^\pi x^2+4x\,dx = \frac{2}{\pi}\left[\frac{1}{3}x^3+2x^2\right]_0^\pi = \frac{2}{\pi}\left(\frac{1}{3}\pi^3+2\pi^2\right) = \frac{2}{3}\pi^2+4\pi$

$a_n = \frac{2}{\pi}\int_0^\pi (x^2+4x)\cos nx\,dx = \frac{2}{\pi}\left[\ \frac{1}{n}(x^2+4x)\sin nx\right]_0^\pi - \frac{2}{\pi n}\int_0^\pi (2x+4)\sin nx\,dx$

$= -\frac{2}{\pi n}\left[-\frac{1}{n}(2x+4)\cos nx\right]_0^\pi - \frac{4}{\pi n^2}\int_0^\pi \cos nx\,dx$

$= \frac{2}{\pi n^2}\left((2\pi+4)(-1)^n-4\right) - \left[\frac{4}{\pi n^3}\sin nx\right]_0^\pi = \frac{4}{\pi n^2}\left(-2+(\pi+2)(-1)^n\right)$ より

$f(x) \sim \frac{1}{3}\pi^2+2\pi + \sum_{n=1}^{\infty}\frac{4}{\pi n^2}\left(-2+(\pi+2)(-1)^n\right)\cos nx$ である.

正弦 $b_n = \frac{2}{\pi}\int_0^\pi (x^2+4x)\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}(x^2+4x)\cos nx\right]_0^\pi + \frac{2}{\pi n}\int_0^\pi (2x+4)\cos nx\,dx$

$= \frac{-2}{\pi n}\left((\pi^2+4\pi)(-1)^n\right) + \frac{2}{\pi n}\left[\ \frac{1}{n}(2x+4)\sin nx\right]_0^\pi - \frac{4}{\pi n^2}\int_0^\pi \sin nx\,dx$

$= \frac{2}{\pi n}(\pi^2+4\pi)(-1)^{n+1} - \frac{4}{\pi n^2}\left[-\frac{1}{n}\cos nx\right]_0^\pi$

$= \frac{2}{\pi n}(\pi^2+4\pi)(-1)^{n+1} + \frac{4}{\pi n^3}\left((-1)^n-1\right)$ より

$f(x) \sim \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}(\pi^2+4\pi)(-1)^{n+1} + \frac{4}{\pi n^3}\left((-1)^n-1\right)\right)\sin nx$ である.

(5) 余弦 $a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}\left[-\cos x\right]_0^\pi = \frac{4}{\pi}$

$a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = \left[\frac{1}{2\pi}\cos 2x\right]_0^\pi = 0$

No.

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin(n+1)x-\sin(n-1)x\,dx$$

$$=\frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x+\frac{1}{n-1}\cos(n-1)x\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left(-\frac{1}{n+1}((-1)^{n+1}-1)+\frac{1}{n-1}((-1)^{n-1}-1)\right)=\frac{1}{\pi}((-1)^{n+1}-1)+\frac{2}{n^2-1}$$

$$=\frac{2}{\pi(n^2-1)}((-1)^{n+1}-1)$$ より

$$f(x)\sim\frac{2}{\pi}+\sum_{n=2}^{\infty}\frac{2}{\pi(n^2-1)}((-1)^{n+1}-1)\cos nx$$ である.

正弦 $f(x)\sim\sin x$.

(6) 余弦 $a_0=\frac{2}{\pi}\int_0^{\pi}e^{2x}dx=\left[\frac{1}{\pi}e^{2x}\right]_0^{\pi}=\frac{1}{\pi}(e^{2\pi}-1)$

$A=\int_0^{\pi}e^{2x}\cos nx\,dx$ とすると.

$$A=\left[\frac{1}{n}e^{2x}\sin nx\right]_0^{\pi}-\frac{2}{n}\int_0^{\pi}e^{2x}\sin nx\,dx$$

$$=-\frac{2}{n}\left[e^{2x}\left(-\frac{1}{n}\cos nx\right)\right]_0^{\pi}-\frac{4}{n^2}\int e^{2x}\cos nx\,dx$$

$$=\frac{2}{n^2}\left(e^{2\pi}(-1)^n-1\right)-\frac{4}{n^2}A \qquad \therefore A=\frac{2}{n^2+4}\left(e^{2\pi}(-1)^n-1\right)$$

$$\therefore a_n=\frac{2}{\pi}A=\frac{4}{\pi(n^2+4)}\left(e^{2\pi}(-1)^n-1\right)$$

$$\therefore f(x)\sim\frac{1}{2\pi}(e^{2\pi}-1)+\sum_{n=1}^{\infty}\frac{4}{\pi(n^2+4)}\left(e^{2\pi}(-1)^n-1\right)\cos nx.$$

正弦. Aの計算から.

$$b_n=\frac{2}{\pi}\int_0^{\pi}e^{2x}\sin nx\,dx=-\frac{n}{\pi}A=\frac{2n}{\pi(n^2+4)}\left(1-e^{2\pi}(-1)^n\right)$$

$$\therefore f(x)\sim\sum_{n=1}^{\infty}\frac{2n}{\pi(n^2+4)}\left(1-e^{2\pi}(-1)^n\right)$$

No. ( )

3.3節

1.(1) $f(x)$ は奇関数 より $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx = \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos nx\, dx$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{\pi n}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = \frac{2}{n}(-1)^{n+1}$$ である

また $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$ より.

$\frac{2}{3}\pi^2 = \sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\right)^2 = \sum_{n=1}^{\infty}\frac{4}{n^2}$ となり. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ を得る.

(2). $f(x)$ は奇関数 より $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = -\frac{2}{\pi n}\left((-1)^n - 1\right)$$ である

また. $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx = 2$ より

nが偶数のときは0になるので奇数のところだけを足している

$2 = \sum_{n=1}^{\infty}\left|-\frac{2}{\pi n}\left((-1)^n - 1\right)\right|^2 = \sum_{n=1}^{\infty}\frac{16}{\pi^2(2n-1)^2}$ となり

$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ となる.

(3). $f(x)$ は偶関数 より $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{3}\pi^2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\, dx = \frac{2}{\pi}\left[\frac{1}{n}x^2\sin nx\right]_0^{\pi} - \frac{4}{\pi n}\int_0^{\pi} x\sin nx\, dx$$

$$= -\frac{4}{\pi n}\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos nx\, dx$$

$$= \frac{4}{n^2}(-1)^n - \frac{4}{\pi n^2}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = \frac{4}{n^2}(-1)^n$$ である.

No.

また、$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{1}{\pi}\left[\frac{1}{5}x^5\right]_{-\pi}^{\pi} = \frac{2}{5}\pi^4$ より

$$\frac{2}{5}\pi^4 = \frac{1}{2}\cdot\frac{4}{9}\pi^4 + \sum_{n=1}^{\infty}\frac{16}{n^4}$$ となる

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{90}\pi^4$ である.

2 (1) $f(x)=2x$ のフーリエ級数は、$f(x)$ が奇関数より $a_n=0$ であり、また、

$$b_n = \frac{4}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{4}{\pi}\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{4}{\pi n}\int_0^{\pi}\cos nx\,dx$$

$$= -\frac{4}{n}(-1)^n + \left[\frac{4}{\pi n^2}\sin nx\right]_0^{\pi} = \frac{4}{n}(-1)^{n+1}$$ より

$S_5(x) = \sum_{n=1}^{5}\frac{4}{n}(-1)^{n+1}\sin nx$ である.

(2) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}1\,dx = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos nx\,dx = \frac{2}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{\pi n}(1-(-1)^n)$$ より

$S_5(x) = 1 + \sum_{n=1}^{5}\frac{2}{\pi n}(1-(-1)^n)\sin nx$ である.

No.

3.4節

1 (1) $f(x)$は奇関数より. $a_n=0$

$$b_n=\frac{2}{l}\int_0^l \sin\frac{\pi n x}{l}dx=\frac{2}{l}\left[\frac{-l}{\pi n}\cos\frac{\pi n x}{l}\right]_0^l$$

$$=-\frac{2}{\pi n}\left((-1)^n-1\right)=\frac{2}{\pi n}\left(1-(-1)^n\right)$$ となり

$$f(x)\sim\sum_{n=1}^{\infty}\frac{2}{\pi n}\left(1-(-1)^n\right)\sin\frac{\pi n}{l}x$$ である

(2) $f(x)$は奇関数より. $a_n=0$

$$b_n=\frac{2}{l}\int_0^l x\sin\frac{\pi n x}{l}dx=\frac{2}{l}\left[-\frac{l}{\pi n}x\cos\frac{\pi n}{l}x\right]_0^l+\frac{2}{\pi n}\int_0^l\cos\frac{\pi n}{l}x\,dx$$

$$=-\frac{2l}{\pi n}(-1)^n+\frac{2}{\pi n}\left[\frac{l}{\pi n}\sin\frac{\pi n}{l}x\right]_0^l=\frac{2l}{\pi n}(-1)^{n+1}$$ となり.

$$f(x)\sim\sum_{n=1}^{\infty}\frac{2l}{\pi n}(-1)^{n+1}\sin\frac{\pi n}{l}x$$ となる.

(3) $g(x)=f(x)-3=-2x$ とすると. (2)より

$$g(x)\sim\sum_{n=1}^{\infty}\frac{4l}{\pi n}(-1)^n\sin\frac{\pi n}{l}x$$ となり

$$f(x)\sim 3+\sum_{n=1}^{\infty}\frac{4l}{\pi n}(-1)^n\sin\frac{\pi n}{l}x$$ となる.

(4) $f(x)$は偶関数より, $b_n=0$

$$a_0=\frac{1}{l}\int_{-l}^{l}x^2dx=\frac{2}{3}l^2$$

$$a_n=\frac{2}{l}\int_0^l x^2\cos\frac{\pi n x}{l}dx=\frac{2}{l}\left[\frac{l}{\pi n}x^2\sin\frac{\pi n}{l}x\right]_0^l-\frac{4}{\pi n}\int_0^l x\sin\frac{\pi n}{l}x\,dx$$

$$=-\frac{4}{\pi n}\left[-\frac{l}{\pi n}x\cos\frac{\pi n}{l}x\right]_0^l-\frac{4l}{\pi^2n^2}\int_0^l\cos\frac{\pi n}{l}x\,dx$$

$$= \frac{4\ell^2}{\pi^2 n^2}(-1)^n - \frac{4\ell}{\pi^2 n^2}\left[\frac{\ell}{\pi n}\sin\frac{\pi n}{\ell}x\right]_0^{\ell} = \frac{4\ell^2}{\pi^2 n^2}(-1)^n$$ より

$$f(x) \sim \frac{1}{3}\ell^2 + \sum_{n=1}^{\infty}\frac{4\ell^2}{\pi^2 n^2}(-1)^n \cos\frac{\pi n}{\ell}x$$ である

(5) 3倍角の公式から

$$f(x) = \frac{3}{4}\sin\frac{\pi x}{\ell} - \frac{1}{4}\sin\frac{3\pi x}{\ell}$$ である.

(6) まず $a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} e^{-x}dx = \frac{1}{\ell}\left[-e^{-x}\right]_{-\ell}^{\ell} = \frac{1}{\ell}(e^{\ell} - e^{-\ell})$ である.

$A = \int_{-\ell}^{\ell} e^{-x}\cos\frac{\pi n x}{\ell}dx$ とすると.

$$A = \left[\frac{\ell}{\pi n}e^{-x}\sin\frac{\pi n x}{\ell}\right]_{-\ell}^{\ell} + \frac{\ell}{\pi n}\int_{-\ell}^{\ell} e^{-x}\sin\frac{\pi n x}{\ell}dx$$

$$= \frac{\ell}{\pi n}\left[-\frac{\ell}{\pi n}e^{-x}\cos\frac{\pi n x}{\ell}\right]_{-\ell}^{\ell} - \frac{\ell^2}{\pi^2 n^2}\int_{-\ell}^{\ell} e^{-x}\cos\frac{\pi n x}{\ell}dx$$

$$= -\frac{\ell^2}{\pi^2 n^2}(-1)^n(e^{-\ell} - e^{\ell}) - \frac{\ell^2}{\pi^2 n^2}A$$ より

$$A = \frac{\ell^2}{\pi^2 n^2 + \ell^2}(-1)^n(e^{\ell} - e^{-\ell})$$ となる

$\therefore a_n = \frac{\ell}{\pi^2 n^2 + \ell^2}(-1)^n(e^{\ell} - e^{-\ell})$ である. また.計算から.

$$b_n = \frac{\pi n}{\ell^2}A = \frac{\pi n}{\pi^2 n^2 + \ell^2}(-1)^n(e^{\ell} - e^{-\ell})$$ となる

$$\therefore f(x) \sim \frac{1}{2\ell}(e^{\ell} - e^{-\ell}) + \sum_{n=1}^{\infty}\frac{e^{\ell} - e^{-\ell}}{\pi^2 n^2 + \ell^2}(-1)^n\left(\ell\cos\frac{\pi n x}{\ell} + \pi n\sin\frac{\pi n x}{\ell}\right)$$

となる.

2.(1) 余弦 $a_0 = \frac{2}{l}\int_0^l 3\ dx = 6$, $a_n = \frac{2}{l}\int_0^l 3\cos\frac{\pi n x}{l}dx = 0$

より $f(x) \sim 3$

正弦 $b_n = \frac{2}{l}\int_0^l 3\cdot\sin\frac{n\pi x}{l}dx = \frac{6}{l}\left[-\frac{l}{\pi n}\cos\frac{\pi n x}{l}\right]_0^l$

$= -\frac{6}{\pi n}\left((-1)^n - 1\right) = \frac{6}{\pi n}\left(1-(-1)^n\right)$ より

$f(x) \sim \sum_{n=1}^{\infty}\frac{6}{\pi n}\left(1-(-1)^n\right)\sin\frac{\pi n x}{l}$ である.

(2) 余弦 $a_0 = \frac{2}{l}\int_0^l x-1\ dx = \frac{2}{l}\left[\frac{1}{2}x^2 - x\right]_0^l = l-2$.

$a_n = \frac{2}{l}\int_0^l (x-1)\cos\frac{\pi n x}{l}dx = \frac{2}{l}\left[\frac{l}{\pi n}(x-1)\sin\frac{\pi n x}{l}\right]_0^l - \frac{2}{\pi n}\int_0^l \sin\frac{\pi n x}{l}dx$

$= \frac{-2}{\pi n}\cdot\left[-\frac{l}{\pi n}\cos\frac{\pi n x}{l}\right]_0^l = \frac{2l}{\pi^2 n^2}\left((-1)^n - 1\right)$ より

$f(x) \sim \frac{l}{2} - 1 + \sum_{n=1}^{\infty}\frac{2l}{\pi^2 n^2}\left((-1)^n - 1\right)\cos\frac{\pi n x}{l}$ である.

正弦 $b_n = \frac{2}{l}\int_0^l (x-1)\sin\frac{\pi n x}{l}dx$

$= \frac{2}{l}\left[-\frac{l}{\pi n}(x-1)\cos\frac{\pi n}{l}x\right]_0^l + \frac{2}{\pi n}\int_0^l \cos\frac{\pi n}{l}x\ dx$

$= -\frac{2}{\pi n}\left((l-1)\cdot(-1)^n + 1\right) + \frac{2}{\pi n}\left[\frac{l}{\pi n}\sin\frac{\pi n}{l}x\right]_0^l$

$= \frac{2}{\pi n}\left((-1)^{n+1}(l-1) - 1\right)$ より

$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{\pi n}\left((-1)^{n+1}(l-1) - 1\right)\sin\frac{\pi n x}{l}$ である

(3) 余弦 $f(x) \sim \cos\frac{2\pi x}{l}$.

正弦 $n=2$ のとき.

$$b_2 = \frac{2}{l}\int_0^l \cos\frac{2\pi x}{l}\sin\frac{2\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{4\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[-\frac{l}{4\pi}\cos\frac{4\pi x}{l}\right]_0^l = 0$$ となる. また $n \neq 2$ のとき.

$$b_n = \frac{2}{l}\int_0^l \cos\frac{2\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{(n+2)\pi x}{l} + \sin\frac{(n-2)\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[-\frac{l}{(n+2)\pi}\cos\frac{(n+2)\pi x}{l} - \frac{l}{(n-2)\pi}\cos\frac{(n-2)\pi x}{l}\right]_0^l$$

$$= -\frac{1}{\pi}\left(\frac{1}{n+2}+\frac{1}{n-2}\right)\left((-1)^n - 1\right) = \frac{2n(1-(-1)^n)}{\pi(n^2-4)}$$ より

$$f(x) \sim \sum_{n\neq 2}\frac{2n(1-(-1)^n)}{\pi(n^2-4)}\sin\frac{\pi n x}{l}$$

(4) 余弦 $a_0 = \frac{2}{l}\int_0^l x^3\,dx = \frac{1}{2}l^3$

$$a_n = \frac{2}{l}\int_0^l x^3\cos\frac{\pi n x}{l}\,dx = \left[\frac{2}{\pi n}x^3\sin\frac{\pi n x}{l}\right]_0^l - \frac{6}{\pi n}\int_0^l x^2\sin\frac{\pi n x}{l}\,dx$$

$$= \frac{6}{\pi n}\left[\frac{l}{\pi n}x^2\cos\frac{\pi n x}{l}\right]_0^l - \frac{12 l}{\pi^2 n^2}\int_0^l x\cos\frac{\pi n x}{l}\,dx$$

$$= \frac{6l^3}{\pi^2 n^2}(-1)^n - \frac{12l}{\pi^2 n^2}\left[\frac{l}{\pi n}x\sin\frac{\pi n x}{l}\right]_0^l + \frac{12l^2}{\pi^3 n^3}\int_0^l \sin\frac{\pi n x}{l}\,dx$$

$$= \frac{6l^3}{\pi^2 n^2}(-1)^n - \frac{12l^2}{\pi^3 n^3}\left[\frac{l}{\pi n}\cos\frac{\pi n x}{l}\right]_0^l$$

$$= \frac{6l^3}{\pi^2 n^2}(-1)^n - \frac{12l^3}{\pi^4 n^4}(-1)^n + \frac{12l^3}{\pi^4 n^4}$$ より.

No.

$$f(x) \sim \frac{1}{4}l^3 + \sum\left(\frac{6l^3}{\pi^2 n^2}(-1)^n - \frac{12l^3}{\pi^4 n^4}(-1)^n + \frac{12l^3}{\pi^4 n^4}\right)\cos\frac{\pi n x}{l}$$ である

正弦 $b_n = \frac{2}{l}\int_0^l x^3 \sin\frac{\pi n x}{l}dx = \frac{2}{l}\left[-\frac{l}{\pi n}x^3\cos\frac{\pi n x}{l}\right]_0^l + \frac{6}{\pi n}\int_0^l x^2\cos\frac{\pi n x}{l}dx$

$$= -\frac{2l^3}{\pi n}(-1)^n + \frac{6}{\pi n}\left[\frac{l}{\pi n}x^2\sin\frac{\pi n x}{l}\right]_0^l - \frac{12l}{\pi^2 n^2}\int_0^l x\sin\frac{\pi n x}{l}dx$$

$$= \frac{2l^3}{\pi n}(-1)^{n+1} - \frac{12l}{\pi^2 n^2}\left[-\frac{l}{\pi n}x\cos\frac{\pi n x}{l}\right]_0^l - \frac{12l^2}{\pi^3 n^3}\int_0^l \cos\frac{\pi n x}{l}dx$$

$$= \frac{2l^3}{\pi n}(-1)^{n+1} + \frac{12l^3}{\pi^3 n^3}(-1)^n$$ より

$$f(x) \sim \sum\left(\frac{2l^3}{\pi n}(-1)^{n+1} + \frac{12l^3}{\pi^3 n^3}(-1)^n\right)\sin\frac{\pi n x}{l}$$ である.

3.5節

1.(1) $\int_{-l}^{l} |1|\, dx = 2l \longrightarrow \infty \quad (l \to \infty)$ より 絶対積分不可能.

(2) $l = 2\pi n + m$ （nは自然数, $0 \leqq m < 2\pi$）とすると

$$\int_{-l}^{l} |\sin 2x|\, dx \geqq \int_{-2\pi n}^{2\pi n} |\sin 2x|\, dx = 4n \cdot \int_{0}^{\frac{\pi}{2}} \sin 2x\, dx$$

$|\sin 2x|$ は周期 $\frac{\pi}{2}$ なので.

$= 4n \cdot \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{2}} = 4n \longrightarrow \infty \quad (n \to \infty)$ より 絶対積分不可能.

(3). $\int_{-l}^{l} |x e^{-x^2}|\, dx = 2 \cdot \int_0^l x e^{-x^2} dx = 2 \cdot \left[-\frac{1}{2} e^{-x^2} \right]_0^l$

$= -e^{-l^2} + 1 \longrightarrow 1 \quad (l \to \infty)$ より 絶対積分可能

(4) $\int_{\frac{1}{l}}^{l} \frac{1}{x^2}\, dx = \left[-\frac{1}{x} \right]_{\frac{1}{l}}^{l} = -\frac{1}{l} + l \longrightarrow \infty \quad (l \to \infty)$

より 絶対積分不可能.

2(1) $f(x)$は偶関数より. $B(u) = 0$.

$A(u) = \int_{-\infty}^{\infty} f(t) \cdot \cos ut\, dt = 2 \cdot \int_0^1 \cos ut\, dt = 2 \left[\frac{1}{u} \sin ut \right]_0^1$

$= \frac{2}{u} \sin u$.

$\therefore f(x) \sim \frac{1}{\pi} \int_0^{\infty} \frac{2}{u} \sin u \cos ux\, du$.

(2). $f(x)$は奇関数 より $A(u) = 0$.

$B(u) = \int_{-\infty}^{\infty} f(t) \sin ut\, dt = 2 \int_0^2 x \sin ut\, dt = \left[-\frac{2}{u} x \cos ut \right]_0^2 + \frac{2}{u} \int_0^2 \cos ut\, dt$

$$= -\frac{4}{u}\cos 2u + \frac{2}{u}\left[\frac{1}{u}\sin ut\right]_0^2 = -\frac{4}{u}\cos 2u + \frac{2}{u^2}\sin 2u$$

$$\therefore f(x) \sim \frac{1}{\pi}\int_0^\infty \left(-\frac{4}{u}\cos 2u + \frac{2}{u^2}\sin 2u\right)\sin ux\, du$$

(3) $A(u) = \int_0^2 (3-2t)\cos ut\, dt = \left[\frac{1}{u}(3-2t)\sin ut\right]_0^2 + \frac{2}{u}\int_0^2 \sin ut\, dt$

$$= -\frac{1}{u}\sin 2u + \frac{2}{u}\left[-\frac{1}{u}\cos ut\right]_0^2 = -\frac{1}{u}\sin 2u + \frac{2}{u^2}(-\cos 2u + 1)$$

$B(u) = \int_0^2 (3-2t)\sin ut\, dt = \left[-\frac{1}{u}(3-2t)\cos ut\right]_0^2 - \frac{2}{u}\int_0^2 \cos ut\, dt$

$$= \frac{1}{u}\cos 2u + \frac{3}{u} - \frac{2}{u}\left[\frac{1}{u}\sin ut\right]_0^2 = \frac{1}{u}\cos 2u + \frac{3}{u} - \frac{2}{u^2}\sin 2u$$ より

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \left(-\frac{1}{u}\sin 2u + \frac{2}{u^2}(-\cos 2u + 1)\right)\cos ux + \left(\frac{1}{u}\cos 2u + \frac{3}{u} - \frac{2}{u^2}\sin 2u\right)\sin ux\, du$$

である

(4) $f(x)$ は偶関数より. $B(u) = 0$

$A(u) = 2\int_0^\pi (\pi - t)\cos ut\, dt = 2\left[(\pi - t)\cdot\frac{1}{u}\sin ut\right]_0^\pi + \frac{2}{u}\int_0^\pi \sin ut\, dt$

$$= \frac{2}{u}\left[-\frac{1}{u}\cos ut\right]_0^\pi = -\frac{2}{u^2}(\cos \pi u - 1)$$ より

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \frac{2}{u^2}(1 - \cos \pi u)\cos ux\, du$$ である.

(5). $A(u) = \int_{-1}^2 (t^2 - 3)\cos ut\, dt = \left[\frac{1}{u}(t^2-3)\sin ut\right]_{-1}^2 - \frac{2}{u}\int_{-1}^2 t\sin ut\, dt$

$$= \frac{1}{u}(\sin 2u - 2\sin u) - \frac{2}{u}\left[t\cdot -\frac{1}{u}\cos ut\right]_{-1}^2 - \frac{2}{u^2}\int_{-1}^2 \cos ut\, dt$$

$$= \frac{1}{u}(\sin 2u - 2\sin u) + \frac{2}{u^2}(2\cos 2u + \cos u) - \frac{2}{u^2}\left[\frac{1}{u}\sin ut\right]_{-1}^{2}$$

$$= \frac{1}{u}(\sin 2u - 2\sin u) + \frac{2}{u^2}(2\cos 2u + \cos u) - \frac{2}{u^3}(\sin 2u + \sin u)$$

$$B(u) = \int_{-1}^{2}(t^2-3)\sin ut\,dt = \left[-\frac{1}{u}(t^2-3)\cos ut\right]_{-1}^{2} + \frac{2}{u}\int_{-1}^{2} t\cos ut\,dt$$

$$= -\frac{1}{u}(\cos 2u + 2\cos u) + \frac{2}{u}\left[\frac{1}{u}t\sin ut\right]_{-1}^{2} - \frac{2}{u^2}\int_{-1}^{2}\sin ut\,dt$$

$$= -\frac{1}{u}(\cos 2u + 2\cos u) + \frac{2}{u^2}(2\sin 2u - \sin u) - \frac{2}{u^2}\left[-\frac{1}{u}\cos ut\right]_{-1}^{2}$$

$$= -\frac{1}{u}(\cos 2u + 2\cos u) + \frac{2}{u^2}(2\sin 2u - \sin u) + \frac{2}{u^3}(\cos 2u - \cos u) \quad \text{より}$$

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \left\{\frac{1}{u}(\sin 2u - 2\sin u) + \frac{2}{u^2}(2\cos 2u + \cos u) - \frac{2}{u^3}(\sin 2u + \sin u)\right\}\cos ux$$
$$+ \left\{-\frac{1}{u}(\cos 2u - 2\cos u) + \frac{2}{u^2}(2\sin 2u - \sin u) + \frac{2}{u^3}(\cos 2u - \cos u)\right\}\sin ux\,du$$

である

(6) $f(x)$は奇関数より　$A(u) = 0$

$$B(u) = 2\int_0^1 t^3\sin ut\,dt = 2\cdot\left[-\frac{1}{u}t^3\cos ut\right]_0^1 + \frac{6}{u}\int_0^1 t^2\cos ut\,dt$$

$$= -\frac{2}{u}\cos u + \frac{6}{u}\left[\frac{1}{u}t^2\sin ut\right]_0^1 - \frac{12}{u^2}\int_0^1 t\sin ut\,dt$$

$$= -\frac{2}{u}\cos u + \frac{6}{u^2}\sin u - \frac{12}{u^2}\left[-\frac{1}{u}t\cos ut\right]_0^1 - \frac{12}{u^3}\int_0^1\cos ut\,dt$$

$$= -\frac{2}{u}\cos u + \frac{6}{u^2}\sin u + \frac{12}{u^3}\cos u - \frac{12}{u^3}\left[\frac{1}{u}\sin ut\right]_0^1$$

$$= -\frac{2}{u}\cos u + \frac{6}{u^2}\sin u + \frac{12}{u^3}\cos u - \frac{12}{u^4}\sin u$$

$$f(x) \sim \frac{1}{\pi}\int_0^\infty\left(-\frac{2}{u}\cos u + \frac{6}{u^2}\sin u + \frac{12}{u^3}\cos u - \frac{12}{u^4}\sin u\right)\sin ux\,du.$$

3.6節

1 (1) 余弦 $C(u)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\int_0^a \cos ut\,dt$

$=\sqrt{\frac{2}{\pi}}\cdot\left[\frac{1}{u}\sin ut\right]_0^a=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{u}\sin au$ となる

正弦 $S(u)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\int_0^a \sin ut\,dt$

$=\sqrt{\frac{2}{\pi}}\left[-\frac{1}{u}\cos ut\right]_0^a=\sqrt{\frac{2}{\pi}}\left(-\frac{1}{u}\cos au+\frac{1}{u}\right)$ である.

(2) $C(u)=\sqrt{\frac{2}{\pi}}\int_0^a (a-t)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[\frac{1}{u}(a-t)\sin ut\right]_0^a+\frac{1}{u}\int_0^a \sin ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{u}\left[-\frac{1}{u}\cos ut\right]_0^a=\sqrt{\frac{2}{\pi}}\frac{1}{u^2}(1-\cos au)$

$S(u)=\sqrt{\frac{2}{\pi}}\int_0^a (a-t)\sin ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[-\frac{1}{u}(a-t)\cos ut\right]_0^a-\frac{1}{u}\int_0^a \cos ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{a}{u}-\frac{1}{u}\left[\frac{1}{u}\sin ut\right]_0^a\right)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{u}-\frac{1}{u^2}\sin au\right)$ である.

(3) $C(u)=\sqrt{\frac{2}{\pi}}\int_0^3 (4t-5)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[\frac{1}{u}(4t-5)\sin ut\right]_0^3-\frac{4}{u}\int_0^3 \sin ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{7}{u}\sin 3u-\frac{4}{u}\left[-\frac{1}{u}\cos ut\right]_0^3\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{7}{u}\sin 3u+\frac{4}{u^2}(\cos 3u-1)\right)$

$S(u)=\sqrt{\frac{2}{\pi}}\int_0^3 (4t-5)\sin ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[-\frac{1}{u}(4t-5)\cos ut\right]_0^3+\frac{4}{u}\int_0^3 \cos ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(-\frac{7}{u}\cos 3u-\frac{5}{u}+\frac{4}{u}\left[\frac{1}{u}\sin ut\right]_0^3\right)$

$=\sqrt{\frac{2}{\pi}}\left(-\frac{7}{u}\cos 3u-\frac{5}{u}+\frac{4}{u^2}\sin 3u\right)$ である

No.

(4) $C(u)=\sqrt{\frac{2}{\pi}}\int_0^1(3t^2+2t-1)\cos ut\,dt$

$=\sqrt{\frac{2}{\pi}}\left(\left[\frac{1}{u}(3t^2+2t-1)\sin ut\right]_0^1-\frac{1}{u}\int_0^1(6t+2)\sin ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{4}{u}\sin u-\frac{1}{u}\left[-\frac{1}{u}(6t+2)\cos ut\right]_0^1-\frac{6}{u^2}\int_0^1\cos ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{4}{u}\sin u+\frac{8}{u^2}\cos u-\frac{2}{u^2}-\frac{6}{u^2}\left[\frac{1}{u}\sin ut\right]_0^1\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{4}{u}\sin u+\frac{8}{u^2}\cos u-\frac{2}{u^2}-\frac{6}{u^3}\sin u\right)$

$S(u)=\sqrt{\frac{2}{\pi}}\int_0^1(3t^2+2t-1)\sin ut\,dt$

$=\sqrt{\frac{2}{\pi}}\left(\left[-\frac{1}{u}(3t^2+2t-1)\cos ut\right]_0^1+\frac{1}{u}\int_0^1(6t+2)\cos ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(-\frac{4}{u}\cos u-\frac{1}{u}+\frac{1}{u}\left[\frac{1}{u}(6t+2)\sin ut\right]_0^1-\frac{6}{u^2}\int_0^1\sin ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(-\frac{4}{u}\cos u-\frac{1}{u}+\frac{8}{u^2}\sin u-\frac{6}{u^2}\left[-\frac{1}{u}\cos ut\right]_0^1\right)$

$=\sqrt{\frac{2}{\pi}}\left(-\frac{4}{u}\cos u-\frac{1}{u}+\frac{8}{u^2}\sin u+\frac{6}{u^3}\cos u-\frac{6}{u^3}\right)$ となる

(5) $a\leq 1$ のとき.

$C(u)=\sqrt{\frac{2}{\pi}}\int_0^a(1-t)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[\frac{1}{u}(1-t)\sin ut\right]_0^a+\frac{1}{u}\int_0^a\sin ut\,dt\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}(1-a)\sin au+\frac{1}{u}\left[-\frac{1}{u}\cos ut\right]_0^a\right)$

$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}(1-a)\sin au-\frac{1}{u^2}(\cos au-1)\right)$

$$S(u)=\sqrt{\frac{2}{\pi}}\int_0^a (1-t)\sin ut\,dt=\sqrt{\frac{2}{\pi}}\left(\left[-\frac{1}{u}(1-t)\cos ut\right]_0^a-\frac{1}{u}\int_0^a \cos ut\,dt\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}(a-1)\cos au+\frac{1}{u}-\frac{1}{u}\left[\frac{1}{u}\sin ut\right]_0^a\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}(a-1)\cos au+\frac{1}{u}-\frac{1}{u^2}\sin au\right)$$

である．また $a\geqq 1$ のときは．

$$C(u)=\sqrt{\frac{2}{\pi}}\int_0^a f(t)\cos ut\,dt=\sqrt{\frac{2}{\pi}}\int_0^1 (1-t)\cos ut\,dt+\sqrt{\frac{2}{\pi}}\int_1^a (t-1)\cos ut\,dt$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u^2}(1-\cos u)+\int_1^a (t-1)\cos ut\,dt\right)$$ ← 上の計算で $a=1$ としたものを使った．

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u^2}(1-\cos u)+\left[\frac{1}{u}(t-1)\sin ut\right]_1^a-\frac{1}{u}\int_1^a \sin ut\,dt\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u^2}(1-\cos u)+\frac{1}{u}(a-1)\sin au-\frac{1}{u}\left[-\frac{1}{u}\cos ut\right]_1^a\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u^2}-\frac{2}{u^2}\cos u+\frac{1}{u}(a-1)\sin au+\frac{1}{u^2}\cos au\right)$$

$$S(u)=\sqrt{\frac{2}{\pi}}\int_0^1 (1-t)\sin ut\,dt+\sqrt{\frac{2}{\pi}}\int_1^a (t-1)\sin ut\,dt$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}-\frac{1}{u^2}\sin u+\left[-\frac{1}{u}(t-1)\cos ut\right]_1^a+\frac{1}{u}\int_1^a \cos ut\,dt\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}-\frac{1}{u^2}\sin u-\frac{1}{u}(a-1)\cos au+\frac{1}{u}\left[\frac{1}{u}\sin ut\right]_1^a\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{u}-\frac{2}{u^2}\sin u-\frac{1}{u}(a-1)\cos au+\frac{1}{u^2}\sin au\right)$$ である

No.

(6) $A=\int_0^a e^{2t}\cos ut\,dt$ とすると.

$$A=\left[\frac{1}{u}e^{2t}\sin ut\right]_0^a-\frac{2}{u}\int_0^a e^{2t}\sin ut\,dt$$

$$=\frac{1}{u}e^{2a}\sin au-\frac{2}{u}\left[-\frac{1}{u}e^{2t}\cos ut\right]_0^a-\frac{4}{u^2}\int_0^a e^{2t}\cos ut\,dt$$

$$=\frac{1}{u}e^{2a}\sin au+\frac{2}{u^2}e^{2a}\cos au-\frac{2}{u^2}-\frac{4}{u^2}A$$

$\therefore A=\frac{1}{u^2+4}\left(ue^{2a}\sin au+2e^{2a}\cos au-2\right)$ となり.

$C(u)=\frac{\sqrt{2}}{\sqrt{\pi}(u^2+4)}\left(ue^{2a}\sin au+2e^{2a}\cos au-2\right)$ となる.

また上の計算から.

$$\int_0^a e^{2t}\sin ut=\frac{u}{2}\left(-A+\frac{1}{u}e^{2a}\sin au\right)$$

$$=\frac{1}{2}\cdot\frac{1}{u^2+4}\left(-u^2e^{2a}\sin au-2ue^{2a}\cos au+2u+(u^2+4)e^{2a}\sin au\right)$$

$=\frac{1}{u^2+4}\left(2e^{2a}\sin au-ue^{2a}\cos au+u\right)$ となり.

$S(u)=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{u^2+4}\left(2e^{2a}\sin au-ue^{2a}\cos au+u\right)$ となる.

2. $C(u)=\sqrt{\frac{2}{\pi}}\int_0^\pi \sin t\cdot\cos ut\,dt=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\int_0^\pi \sin(1+u)t+\sin(1-u)t\,dt$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{1}{1+u}\cos(1+u)t+\frac{1}{u-1}\cos(u-1)t\right]_0^\pi$$

$=\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{1+u}+\frac{1}{u-1}\right)\left(\cos(1+u)\pi-1\right)=\sqrt{\frac{2}{\pi}}\frac{1}{1-u^2}\left(\cos\pi u+1\right)$ より

No.

$$f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{1}{1-u^2}(\cos\pi u+1)\cos ux\,du$$ となり.

$$\int_0^\infty \frac{1}{1-u^2}(\cos\pi u+1)\cos ux\,du = \begin{cases} \frac{\pi}{2}|\sin x| & (|x| \leqq \pi) \\ 0 & \text{その他} \end{cases}$$ となる

3. $A = \int_0^\infty e^{-t}\sin ut\,dt$ とすると

$$A = \left[-e^{-t}\sin ut\right]_0^\infty + u\int_0^\infty e^{-t}\cos ut\,dt$$

$$= \left[-ue^{-t}\cos ut\right]_0^\infty - u^2\int_0^\infty e^{-t}\sin ut\,dt$$

$$= u - u^2 A$$

より $A = \frac{u}{u^2+1}$ である これより

$$f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{u}{u^2+1}\sin ux\,du$$ となり

$$\int_0^\infty \frac{u}{u^2+1}\sin ux = \frac{\pi}{2}e^{-x}$$ を得る.

3.7節

1(1) $c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{2\pi}\int_0^{\pi} 2\,dx=1$

$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx=\frac{1}{2\pi}\int_0^{\pi}2e^{-inx}dx=\frac{1}{\pi}\left[-\frac{1}{in}e^{-inx}\right]_0^{\pi}$

$=\frac{i}{\pi n}((-1)^n-1)$ より

$f(x)\sim 1+\sum_{n\neq 0}\frac{i}{\pi n}((-1)^n-1)e^{inx}$

(2) $c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx=\frac{1}{\pi}\int_0^{\pi}x\,dx=\frac{1}{2}\pi$

$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{0}-xe^{-inx}dx+\frac{1}{2\pi}\int_0^{\pi}xe^{-inx}dx$

$=\frac{1}{2\pi}\left[\frac{1}{in}xe^{-inx}\right]_{-\pi}^{0}-\int_{-\pi}^{0}\frac{1}{2\pi in}e^{-inx}dx$

$+\frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx}\right]_0^{\pi}+\frac{1}{2\pi in}\int_0^{\pi}e^{-inx}dx$

$=\frac{1}{2\pi in}\pi(-1)^n-\left[\frac{1}{2\pi n^2}e^{-inx}\right]_{-\pi}^{0}-\frac{1}{2\pi in}\pi(-1)^n+\frac{1}{2\pi n^2}\left[e^{-inx}\right]_0^{\pi}$

$=\frac{1}{2\pi n^2}(-1+(-1)^n+(-1)^n-1)=\frac{1}{\pi n^2}((-1)^n-1)$

$\therefore f(x)\sim\frac{1}{2}\pi+\sum_{n\neq 0}\frac{1}{\pi n^2}((-1)^n-1)e^{inx}$

(3) $\sin 3x=\frac{e^{3ix}-e^{-3ix}}{2i}$ より $f(x)\sim\frac{1}{2i}(e^{3ix}-e^{-3ix})$

(4) $c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^3dx=0$

$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^3e^{-inx}dx=\frac{1}{2\pi}\left[-\frac{1}{in}x^3e^{-inx}\right]_{-\pi}^{\pi}+\frac{3}{2\pi in}\int_{-\pi}^{\pi}x^2e^{-inx}dx$

No.

$$= \frac{i}{2\pi n}\left(\pi^3(-1)^n - (-\pi)^3(-1)^n\right) + \frac{3}{2\pi n^2}\left[x^2 e^{-inx}\right]_{-\pi}^{\pi} - \frac{3}{\pi n^2}\int_{-\pi}^{\pi} x e^{-inx}\,dx$$

$$= \frac{i\pi^2}{n}(-1)^n + \frac{3}{i\pi n^3}\left[x e^{-inx}\right]_{-\pi}^{\pi} - \frac{3}{i\pi n^3}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$= \frac{i\pi^2}{n}(-1)^n - \frac{3i}{\pi n^3}\left(\pi(-1)^n + \pi(-1)^n\right) - \frac{3}{\pi n^4}\left[e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \frac{i\pi^2}{n}(-1)^n - \frac{6i}{n^3}(-1)^n$$ より

$$f(x) \sim \sum_{n\neq 0}\left(\frac{i\pi^2}{n}(-1)^n - \frac{6i}{n^3}(-1)^n\right)e^{inx}$$

2(1) $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx = \frac{1}{2\pi}\left[-\frac{1}{in} x e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{2\pi i n}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$= \frac{i}{2\pi n}\left(\pi(-1)^n + \pi(-1)^n\right) + \frac{1}{2\pi n^2}\left[e^{-inx}\right]_{-\pi}^{\pi} = \frac{i}{n}(-1)^n$$ より

$$f(x) \sim \sum_{n\neq 0}\frac{i}{n}(-1)^n e^{inx}.$$ である. また.

$$f(x) \sim \sum_{n=1}^{\infty}\frac{i}{n}(-1)^n e^{inx} + \sum_{n=1}^{\infty}\frac{i}{(-n)}(-1)^{(-n)}\cdot e^{i(-n)x}$$

$$= \sum_{n=1}^{\infty}\frac{i}{n}(-1)^n\cdot\left(e^{inx} - e^{-inx}\right)$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}2\cdot\frac{e^{inx}-e^{-inx}}{2i} = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$ となる.

No.

(2) $C_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2-x+1\,dx=\frac{1}{\pi}\int_0^{\pi}x^2+1\,dx=\frac{1}{\pi}\left[\frac{1}{3}x^3+x\right]_0^{\pi}=\frac{\pi^2}{3}+1.$

$$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2-x+1)e^{-inx}dx=\frac{1}{2\pi}\left[-\frac{1}{in}(x^2-x+1)e^{-inx}\right]_{-\pi}^{\pi}+\frac{1}{2\pi in}\int_{-\pi}^{\pi}(2x-1)e^{-inx}dx$$

$$=\frac{i}{2\pi n}(-1)^n\left((\pi^2-\pi+1)-(\pi^2+\pi+1)\right)$$

$$+\frac{1}{2\pi n^2}\left[(2x-1)e^{-inx}\right]_{-\pi}^{\pi}-\frac{1}{\pi n^2}\int_{-\pi}^{\pi}e^{-inx}dx$$

$$=\frac{i}{n}(-1)^{n+1}+\frac{1}{2\pi n^2}(-1)^n\left(2\pi-1-(-2\pi-1)\right)+\frac{1}{\pi in^3}\left[e^{-inx}\right]_{-\pi}^{\pi}$$

$$=\frac{i}{n}(-1)^{n+1}+\frac{2}{n^2}(-1)^n$$ より

$$f(x)\sim1+\frac{\pi^2}{3}+\sum_{n\neq0}\left(\frac{i}{n}(-1)^{n+1}+\frac{2}{n^2}(-1)^n\right)e^{inx}$$ である. また

$$f(x)\sim1+\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{i}{n}(-1)^{n+1}+\frac{2}{n^2}(-1)^n\right)e^{inx}+\sum_{n=1}^{\infty}\left(\frac{i}{-n}(-1)^{n+1}+\frac{2}{n^2}(-1)^n\right)e^{-inx}$$

$$=1+\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cdot\frac{e^{inx}+e^{-inx}}{2}+\frac{2}{n}(-1)^n\frac{e^{inx}-e^{-inx}}{2i}$$

$$=1+\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx+\frac{2}{n}(-1)^n\sin nx$$ である.

(3) $f(x)-1=g(x)$ とすると. $g(x)=\begin{cases}0 & -\pi<x<0\\ x & 0\leqq x\leqq\pi\end{cases}$ である.

$g(x)$のフーリエ級数は.

$$C_0=\frac{1}{2\pi}\int_0^{\pi}x\,dx=\frac{1}{4}\pi.$$

$$C_n=\frac{1}{2\pi}\int_0^{\pi}xe^{-inx}dx=\frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx}\right]_0^{\pi}+\frac{1}{2\pi in}\int_0^{\pi}e^{-inx}dx$$

No.

$$= \frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}\left[e^{-inx}\right]_0^{\pi} = \frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}\left((-1)^n - 1\right)$$ より

$$g(x) \sim \frac{1}{4}\pi + \sum_{n\neq 0}\left(\frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}((-1)^n-1)\right)e^{inx}$$

$$\therefore f(x) \sim 1 + \frac{1}{4}\pi + \sum_{n\neq 0}\left(\frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}((-1)^n-1)\right)e^{inx}$$ である. また

$$f(x) \sim 1 + \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left(\frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}((-1)^n-1)\right)e^{inx} + \sum_{n=1}^{\infty}\left(-\frac{i}{2n}(-1)^n + \frac{1}{2\pi n^2}((-1)^n-1)\right)e^{-inx}$$

$$= 1 + \frac{1}{4}\pi + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{e^{inx}-e^{-inx}}{2i} + \frac{1}{\pi n^2}((-1)^n-1)\frac{e^{inx}+e^{-inx}}{2}$$

$$= 1 + \frac{1}{4}\pi + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx + \frac{1}{\pi n^2}((-1)^n-1)\cos nx$$ である.

(4) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx$$

$$= \frac{1}{2\pi}\cdot\frac{1}{1-in}\left[e^{(1-in)x}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{1}{1-in}\left(e^{\pi-in\pi} - e^{-\pi+in\pi}\right)$$

$$= \frac{1}{2\pi(1-in)}(-1)^n\cdot(e^{\pi}-e^{-\pi})$$ より

$$f(x) \sim \sum \frac{1}{2\pi(1-in)}(-1)^n(e^{\pi}-e^{-\pi})e^{inx}$$ となる. また.

$$f(x) \sim \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left(\sum\frac{1+in}{2(1+n^2)}(-1)^n e^{inx}\right)$$

$$= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1+in}{2(1+n^2)}(-1)^n e^{inx} + \sum_{n=1}^{\infty}\frac{1-in}{2(1+n^2)}(-1)^n e^{-inx}\right)$$

$$= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cdot\frac{e^{inx}+e^{-inx}}{2} - \frac{n}{1+n^2}(-1)^n\cdot\frac{e^{inx}-e^{-inx}}{2i}\right)$$

$$= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx - \frac{n}{n^2+1}(-1)^n\sin nx\right)$$ である.

No.

3.8節

1.(1) $\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-iut}dt=\frac{1}{\sqrt{2\pi}}\int_{-4}^{4}(x-4)e^{-iut}dt$

$=\frac{1}{\sqrt{2\pi}}\left(\left[-\frac{1}{iu}(x-4)e^{-iut}\right]_{-4}^{4}+\frac{1}{iu}\int_{-4}^{4}e^{-iut}dt\right)$

$=\frac{1}{\sqrt{2\pi}}\left(\frac{8i}{u}e^{4iu}+\left[\frac{1}{u^2}e^{-iut}\right]_{-4}^{4}\right)$

$=\frac{1}{\sqrt{2\pi}}\left(\frac{8i}{u}e^{4iu}+\frac{1}{u^2}e^{-4iu}+\frac{1}{u^2}e^{4iu}\right)$ となる

(2) $\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}|x|e^{-iut}dt=\frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{0}-xe^{-iut}dt+\int_{0}^{1}xe^{-iut}dt\right)$

$=\frac{1}{\sqrt{2\pi}}\left(\left[\frac{1}{iu}xe^{-iut}\right]_{-1}^{0}-\frac{1}{iu}\int_{-1}^{0}e^{-iut}dt+\left[-\frac{1}{iu}xe^{-iut}\right]_{0}^{1}+\frac{1}{iu}\int_{0}^{1}e^{-iut}dt\right)$

$=\frac{1}{\sqrt{2\pi}}\left(-\frac{i}{u}e^{iu}-\frac{1}{u^2}\left[e^{-iut}\right]_{-1}^{0}+\frac{i}{u}e^{-iu}+\frac{1}{u^2}\left[e^{-iut}\right]_{0}^{1}\right)$

$=\frac{1}{\sqrt{2\pi}}\left(-\frac{i}{u}e^{iu}-\frac{2}{u^2}+\frac{1}{u^2}e^{iu}+\frac{i}{u}e^{-iu}+\frac{1}{u^2}e^{-iu}\right)$ である.

(3) $\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-t}e^{-iut}dt=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-(1+iu)t}dt$

$=\frac{1}{\sqrt{2\pi}}\left[-\frac{1}{1+iu}e^{-(1+iu)t}\right]_{0}^{\infty}=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+iu}$ である

(4) $\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{0}^{1}x^3e^{-iut}dt=\frac{1}{\sqrt{2\pi}}\left(\left[-\frac{1}{iu}x^3e^{-iut}\right]_{0}^{1}+\frac{3}{iu}\int_{0}^{1}x^2e^{-iut}dt\right)$

$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u}e^{-iu}+\left[\frac{3}{u^2}x^2e^{-iut}\right]_{0}^{1}-\frac{6}{u^2}\int_{0}^{1}xe^{-iut}dt\right)$

$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u}e^{-iu}+\frac{3}{u^2}e^{-iu}+\frac{6}{iu^3}\left[xe^{-iut}\right]_{0}^{1}-\frac{6}{iu^3}\int_{0}^{1}e^{-iut}dt\right)$

No.

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u}e^{-iu}+\frac{3}{u^2}e^{-iu}-\frac{6i}{u^3}e^{-iu}-\frac{6}{u^4}\left[e^{-iut}\right]_0^1\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u}e^{-iu}+\frac{3}{u^2}e^{-iu}-\frac{6i}{u^3}e^{-iu}-\frac{6}{u^4}e^{-iu}+\frac{6}{u^4}\right)$$ である

2(1) $$\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{-2}^{2}2e^{-iut}dt=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{iu}e^{-iut}\right]_{-2}^{2}$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{2i}{u}\left(e^{-2iu}-e^{2iu}\right)$$ である. また.

$$f(x)\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(u)e^{iux}du=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2i}{u}\left(e^{-2iu}-e^{2iu}\right)e^{iux}du$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2}{u}\frac{e^{2iu}-e^{-2iu}}{2i}\cdot(\cos ux+i\sin ux)\,du$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2}{u}\cdot\sin 2u\,(\cos ux+i\sin ux)\,du$$

$$=\frac{4}{\pi}\int_{0}^{\infty}\frac{1}{u}\sin 2u\cdot\cos ux\,du$$ となる.

(2) $$\hat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}te^{-iut}dt=\frac{1}{\sqrt{2\pi}}\left(\left[-\frac{1}{iu}te^{-iut}\right]_{-1}^{1}+\frac{1}{iu}\int_{-1}^{1}e^{-iut}dt\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{i}{u}\left(e^{-iu}+e^{iu}\right)+\frac{1}{u^2}\left[e^{-iut}\right]_{-1}^{1}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{i}{u}\left(e^{-iu}+e^{iu}\right)+\frac{1}{u^2}\left(e^{-iu}-e^{iu}\right)\right)$$ である. また.

$$f(x)\sim\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{i}{u}\left(e^{-iu}+e^{iu}\right)+\frac{1}{u^2}\left(e^{-iu}-e^{iu}\right)\right)e^{iux}du$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{i}{u}\cdot\cos u-\frac{i}{u^2}\sin u\right)(\cos ux+i\sin ux)\,du$$

No.

$$= \frac{2}{\pi}\int_0^\infty \left(-\frac{1}{u}\cos u + \frac{1}{u^2}\sin u\right)\sin ux \, du$$ となる．

(3) $$\hat{f}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|t|}e^{-iut}\,dt = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{(1-iu)t}\,dt + \int_0^\infty e^{(-1-iu)t}\,dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\left[\frac{1}{1-iu}e^{(1-iu)t}\right]_{-\infty}^{0} + \left[\frac{-1}{1+iu}e^{-(1+iu)t}\right]_0^\infty\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-iu} + \frac{1}{1+iu}\right) = \frac{1}{\sqrt{2\pi}}\cdot\frac{2}{1+u^2}$$ である．また．

$$f(x) \sim \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}e^{iux}\,du = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}(\cos ux + i\sin ux)\,du$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2}{1+u^2}\cos ux \, du$$ となる．

(4) $$\hat{f}(u) = \frac{1}{\sqrt{2\pi}}\int_{-3}^{3}(t^2-1)e^{-iut}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left(\left[-\frac{1}{iu}(t^2-1)e^{-iut}\right]_{-3}^{3} + \frac{2}{iu}\int_{-3}^{3} t e^{-iut}\,dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{i}{u}\left(8e^{-3iu} - 8e^{3iu}\right) + \left[\frac{2}{u^2}te^{-iut}\right]_{-3}^{3} - \frac{2}{u^2}\int_{-3}^{3}e^{-iut}\,dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{8i}{u}(e^{-3iu} - e^{3iu}) + \frac{6}{u^2}(e^{-3iu} + e^{3iu}) + \frac{2}{iu^3}\left[e^{-iut}\right]_{-3}^{3}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{8i}{u}(e^{-3iu} - e^{3iu}) + \frac{6}{u^2}(e^{-3iu} + e^{3iu}) - \frac{2i}{u^3}(e^{-3iu} - e^{3iu})\right)$$ となる．また

$$f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{16}{u}\sin 3u + \frac{12}{u^2}\cos 3u - \frac{4}{u^3}\sin 3u\right)(\cos ux + i\sin ux)\,du$$

$$= \frac{1}{\pi}\int_0^\infty\left(\frac{16}{u}\sin 3u + \frac{12}{u^2}\cos 3u - \frac{4}{u^3}\sin 3u\right)\cos ux \, du.$$

No.

3.9節

1(1) $f(x)=1$ の正弦級数のフーリエ係数 A_n は.

$$A_n=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx=\frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}=-\frac{2}{\pi n}\left((-1)^n-1\right)$$ より

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{\pi n}\left(1-(-1)^n\right)e^{-k^2n^2t}\sin nx$$ である.

(2). $u(x,0)$ の正弦級数のフーリエ係数 A_n は

$$A_n=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}-\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}\sin nx\,dx\right)$$

$$=\frac{2}{\pi}\left(\left[\frac{1}{n}\cos nx\right]_0^{\frac{\pi}{2}}+\left[-\frac{1}{n}\cos nx\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$=\frac{2}{\pi}\left(\frac{1}{n}\left(\cos\frac{\pi n}{2}-1\right)-\frac{1}{n}\left((-1)^n-\cos\frac{\pi n}{2}\right)\right)$$

$$=\frac{2}{\pi n}\left((-1)^{n+1}-1+2\cos\frac{\pi n}{2}\right)$$ より

$$u(x,t)=\sum_{n=1}^{2}\frac{2}{\pi n}\left((-1)^{n+1}-1+2\cos\frac{\pi n}{2}\right)e^{-k^2n^2t}\cdot\sin nx$$ である

2(1). $f(x)$ に対して.

$$\int_0^{\pi}f(x)\sin nx\,dx=\int_0^{\pi}x\sin nx\,dx$$

$$=\left[-\frac{1}{n}x\cos nx\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx=\frac{\pi}{n}(-1)^{n+1}$$ より

$$u(x,y)=\sum\frac{2}{n}\frac{(e^{ny}-e^{-ny})}{(e^{n\pi}-e^{-n\pi})}(-1)^{n+1}\sin nx$$ である.

No.

(2) $f(x) = 1 - 2\cdot\frac{1-\cos 2x}{2} = \cos 2x$ より

$n=2$ のとき.

$$\int_0^\pi \cos 2x\cdot\sin 2x\,dx = \frac{1}{2}\int_0^\pi \sin 4x\,dx = 0.$$

$n \neq 2$ のとき

$$\int_0^\pi \cos 2x\sin nx\,dx = \frac{1}{2}\int_0^\pi \sin(n+2)x + \sin(n-2)x\,dx$$

$$= \frac{1}{2}\left[\frac{1}{n+2}\cos(n+2)x - \frac{1}{n-2}\cos(n-2)x\right]_0^\pi$$

$$= \frac{1}{2}\left(-\left(\frac{1}{n+2}+\frac{1}{n-2}\right)\left((-1)^n-1\right)\right) = \frac{n}{n^2-4}\left(1-(-1)^n\right)$$ である

$$\therefore u(x,t) = \sum_{n\neq 2}\frac{2(e^{ny}-e^{-ny})}{\pi(e^{n\pi}-e^{-n\pi})}\cdot\frac{n}{n^2-4}\left(1-(-1)^n\right)\sin nx$$

3. $u(x,y) = X(x)\cdot Y(y)$ として教科書と同様にして解くと.

$X(x) = C_n\cdot\sin nx$ がわかる. このとき. $Y''(y) = n^2\,Y(y)$ であるが.

初期条件と境界条件が異なるので. 補題 3.9.3 は使えない. そこでこれを解くと.

特性方程式.

$\lambda^2 = n^2$ の解は $\pm n$ より. $Y(y) = a\cdot e^{ny} + b e^{-ny}$ となるが.

$u(x,\pi) = 0$ より $Y(\pi) = 0$ となり. $a e^{\pi n} + b e^{-\pi n} = 0$ である. これより

$Y(y) = D_n\cdot\left(e^{n(\pi-y)} - e^{-n(\pi-y)}\right)$ ($a = -D_n\cdot e^{-\pi n}$, $b = D_n e^{\pi n}$ とした)

となる. $\therefore$ $u(x,y) = \sum_{n=1}^{\infty} A_n\left(e^{n(\pi-y)} - e^{-n(\pi-y)}\right)\sin nx$ となるが.

$u(x,0) = f(x)$ より. フーリエ級数を考えれば.

No.

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{\pi}\cdot\frac{e^{n(\pi-y)}-e^{-n(\pi-y)}}{e^{\pi n}-e^{-\pi n}}\left(\int_0^{\pi}f(t)\sin nt\,dt\right)\sin nx$$ となる.

また. $f(x)=x$ について.

$$\int_0^{\pi}x\sin nx\,dx=\left[-\frac{1}{n}x\cos nx\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx=\frac{\pi}{n}(-1)^{n+1}$$ より.

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{n}\frac{e^{n(\pi-y)}-e^{-n(\pi-y)}}{e^{\pi n}-e^{-\pi n}}(-1)^{n+1}\sin nx$$ である