問題2.3 $\boxed{1}$

(1) $\dfrac{1}{2s+1}=\dfrac{1}{2}\cdot\dfrac{1}{s+1/2}=\dfrac{1}{2}\mathcal{L}(e^{-\frac{t}{2}})=\mathcal{L}(\frac{1}{2}e^{-\frac{t}{2}})\quad\therefore \mathcal{L}^{-1}\left(\dfrac{1}{2s+1}\right)=\dfrac{1}{2}e^{-\frac{t}{2}}$

(2) $\dfrac{2s}{(s-2)^2}=\dfrac{2(s-2)+4}{(s-2)^2}=\dfrac{2}{s-2}+\dfrac{4}{(s-2)^2}=2\mathcal{L}(e^{2t})+4\mathcal{L}(t)(s-2)$

$=\mathcal{L}(2e^{2t})+\mathcal{L}(4te^{2t})=\mathcal{L}(2e^{2t}(1+2t))\quad\therefore \mathcal{L}^{-1}\left(\dfrac{2s}{(s-2)^2}\right)=2e^{2t}(1+2t)$

(3) $\dfrac{s}{s^2-2s+5}=\dfrac{(s-1)+1}{(s-1)^2+4}=\dfrac{s-1}{(s-1)^2+4}+\dfrac{1}{2}\cdot\dfrac{2}{(s-1)^2+4}$

$=\mathcal{L}(\cos 2t)(s-1)+\frac{1}{2}\mathcal{L}(\sin 2t)(s-1)=\mathcal{L}(e^t\cos 2t)+\frac{1}{2}(e^t\sin 2t)=\mathcal{L}\left(\frac{1}{2}e^t(2\cos 2t+\sin 2t)\right)$

$\therefore \mathcal{L}^{-1}\left(\dfrac{s}{s^2-2s+5}\right)=\dfrac{1}{2}e^t(2\cos 2t+\sin 2t)$

(4) $\dfrac{s}{s^2-2s-3}=\dfrac{s}{(s-1)^2-4}=\dfrac{(s-1)+1}{(s-1)^2-4}=\dfrac{s-1}{(s-1)^2-4}+\dfrac{1}{2}\cdot\dfrac{2}{(s-1)^2-4}$

$=\mathcal{L}(\cosh 2t)(s-1)+\frac{1}{2}\mathcal{L}(\sinh 2t)(s-1)=\mathcal{L}(e^t\cosh 2t)+\frac{1}{2}\mathcal{L}(e^t\sinh 2t)$

$=\mathcal{L}\left(\frac{1}{2}e^t(2\cosh 2t+\sinh 2t)\right)\quad\therefore \mathcal{L}^{-1}\left(\dfrac{s}{s^2-2s-3}\right)=\dfrac{1}{2}e^t(2\cosh 2t+\sinh 2t)=\dfrac{1}{4}(3e^{3t}+e^{-t})$

(5) $\dfrac{e^{-2s}}{(s+2)^3}=e^{-2s}\mathcal{L}\left(\dfrac{t^2}{2}\right)(s+2)=e^{-2s}\mathcal{L}\left(\dfrac{t^2}{2}e^{-2t}\right)=\left(\dfrac{t^2}{2}e^{-2t}\right)_2$

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(6) $\dfrac{e^{-s}}{s^2-4}=e^{-s}\left(\dfrac{1}{2}\cdot\dfrac{2}{s^2-4}\right)=\dfrac{e^{-s}}{2}\mathcal{L}(\sinh 2t)=e^{-s}\mathcal{L}\left(\dfrac{1}{2}\sinh 2t\right)=\left(\dfrac{1}{2}\sinh 2t\right)_1$

$\boxed{2}$ (1) $\dfrac{s-8}{s^2-s-6}=\dfrac{s-8}{(s+2)(s-3)}=\dfrac{2}{s+2}-\dfrac{1}{s-3}=2\mathcal{L}(e^{-2t})-\mathcal{L}(e^{3t})$

$=\mathcal{L}(2e^{-2t}-e^{3t})\quad\therefore \mathcal{L}^{-1}\left(\dfrac{s-8}{s^2-s-6}\right)=2e^{-2t}-e^{3t}$

(2) $\dfrac{1}{(s^2+1)(s^2+4)}=\dfrac{1}{3}\left(\dfrac{1}{s^2+1}-\dfrac{1}{s^2+4}\right)=\dfrac{1}{3}\left(\mathcal{L}(\sin t)-\dfrac{1}{2}\mathcal{L}(\sin 2t)\right)$

$=\mathcal{L}\left(\frac{1}{3}\sin t-\frac{1}{6}\sin 2t\right)\quad\therefore \mathcal{L}^{-1}\left(\dfrac{1}{(s^2+1)(s^2+4)}\right)=\dfrac{1}{3}\sin t-\dfrac{1}{6}\sin 2t$

(3) $\dfrac{2s^2-3s-2}{s(s+2)(s-1)} = \dfrac{1}{s} + \dfrac{2}{s+2} - \dfrac{1}{s-1} = \mathcal{L}(1) + 2\mathcal{L}(e^{-2t}) - \mathcal{L}(e^{t})$

$= \mathcal{L}(1+2e^{-2t}-e^{t}) \quad \therefore \mathcal{L}^{-1}\left(\dfrac{2s^2-3s-2}{s(s+2)(s-1)}\right) = \underline{1+2e^{-2t}-e^{t}}$

(4) $\dfrac{3s^2+10s+10}{(s+2)(s^2+2s+2)} = \dfrac{1}{s+2} + \dfrac{2s+4}{s^2+2s+2} = \mathcal{L}(e^{-2t}) + \dfrac{2(s+1)+2}{(s+1)^2+1}$

$= \mathcal{L}(e^{-2t}) + 2\cdot\dfrac{s+1}{(s+1)^2+1} + 2\cdot\dfrac{1}{(s+1)^2+1} = \mathcal{L}(e^{-2t}) + 2\mathcal{L}(\cos t)(s+1) + 2\mathcal{L}(\sin t)(s+1)$

$= \mathcal{L}(e^{-2t}) + 2\mathcal{L}(e^{-t}\cos t) + 2\mathcal{L}(e^{-t}\sin t) = \mathcal{L}(e^{-2t} + 2e^{-t}(\cos t + \sin t))$

$\therefore \mathcal{L}^{-1}\left(\dfrac{3s^2+10s+10}{(s+2)(s^2+2s+2)}\right) = \underline{e^{-2t} + 2e^{-t}(\cos t + \sin t)}$

(5) $\dfrac{1}{s(s^2-4)} = \dfrac{1}{4}\left(\dfrac{s}{s^2-4} - \dfrac{1}{s}\right) = \dfrac{1}{4}\left(\mathcal{L}(\cosh 2t) - \mathcal{L}(1)\right)$

$= \mathcal{L}\left(\dfrac{1}{4}(\cosh 2t - 1)\right) \quad \therefore \mathcal{L}^{-1}\left(\dfrac{1}{s(s^2-4)}\right) = \dfrac{1}{4}(\cosh 2t - 1)$

(6) $\dfrac{1}{s^2(s-1)^3} = \dfrac{As+B}{s^2} + \dfrac{Cs^2+Ds+E}{(s-1)^3}$ とおいて、分母を払うと

$1 = (As+B)(s-1)^3 + (Cs^2+Ds+E)s^2$

$= (A+C)s^4 + (-3A+B+D)s^3 + (3A-3B+E)s^2 + (-A+3B)s - B$

$\therefore \begin{cases} A+C=0 \\ -3A+B+D=0 \\ 3A-3B+E=0 \\ -A+3B=0 \\ -B=1 \end{cases} \qquad \therefore A=-3,\ B=-1,\ C=3,\ D=-8,\ E=6$

$\therefore \dfrac{1}{s^2(s-1)^3} = \dfrac{-3s-1}{s^2} + \dfrac{3s^2-8s+6}{(s-1)^3}$

$= -3\cdot\dfrac{1}{s} - \dfrac{1}{s^2} + \dfrac{3(s^2-2s+1)-2(s-1)+1}{(s-1)^3} = -3\mathcal{L}(1) - \mathcal{L}(t) + \dfrac{3}{s-1} - \dfrac{2}{(s-1)^2} + \dfrac{1}{(s-1)^3}$

$$= \mathcal{L}(-3-t)+3\mathcal{L}(e^{t})-2\mathcal{L}(t)(s-1)+\frac{1}{2}\mathcal{L}(t^2)(s-1)$$

$$= \mathcal{L}(-3-t+3e^{t})-2\mathcal{L}(te^{t})+\frac{1}{2}(t^2e^{t})=\mathcal{L}\left(-3-t+e^{t}\left(3-2t+\frac{t^2}{2}\right)\right)$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^3}\right)=-3-t+e^{t}\left(3-2t+\frac{t^2}{2}\right)$$

$\boxed{3}$ (1) $\displaystyle \frac{1}{s^2(s+1)}=\frac{1}{s^2}\cdot\frac{1}{s+1}=\mathcal{L}(t)\cdot\mathcal{L}(e^{-t})=\mathcal{L}(t*e^{-t})$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)}\right)=t*e^{-t}=\int_0^t (t-u)e^{-u}du=t\int_0^t e^{-u}du-\int_0^t ue^{-u}du$$

$$=t\left[-e^{-u}\right]_0^t-\left\{\left[-ue^{-u}\right]_0^t-\int_0^t(-e^{-u})du\right\}=-t(e^{-t}-1)-\left\{\left[-ue^{-u}\right]_0^t-\int_0^t(-e^{-u})du\right\}$$

$$=-t(e^{-t}-1)-\left\{-te^{-t}+\left[-e^{-u}\right]_0^t\right\}=\cancel{-te^{-t}}+t+\cancel{te^{-t}}-(e^{-t}-1)=t+e^{-t}-1$$

(2) $\displaystyle \frac{1}{(s+1)^2(s-2)}=\mathcal{L}(t)(s+1)\cdot\mathcal{L}(e^{2t})=\mathcal{L}(te^{-t})\cdot\mathcal{L}(e^{2t})=\mathcal{L}(te^{-t}*e^{2t})$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2(s-2)}\right)=te^{-t}*e^{2t}=\int_0^t e^{2(t-u)}\cdot ue^{-u}du=\int_0^t e^{2t}\cdot ue^{-3u}du$$

$$=e^{2t}\int_0^t ue^{-3u}du=e^{2t}\left\{\left[u\left(-\frac{1}{3}e^{-3u}\right)\right]_0^t-\int_0^t\left(-\frac{1}{3}e^{-3u}\right)du\right\}$$

$$=e^{2t}\left\{-\frac{t}{3}e^{-3t}+\frac{1}{3}\left[-\frac{1}{3}e^{-3u}\right]_0^t\right\}=-\frac{t}{3}e^{-t}-\frac{e^{2t}}{9}(e^{-3t}-1)$$

$$=-\frac{t}{3}e^{-t}-\frac{1}{9}e^{-t}+\frac{e^{2t}}{9}$$

(3) $\displaystyle \frac{s^2}{(s^2+4)^2}=\frac{s}{s^2+4}\cdot\frac{s}{s^2+4}=\mathcal{L}(\cos 2t)\cdot\mathcal{L}(\cos 2t)=\mathcal{L}(\cos 2t*\cos 2t)$

$$\therefore \mathcal{L}^{-1}\left(\frac{s^2}{(s^2+4)^2}\right)=\cos 2t*\cos 2t=\int_0^t \cos 2(t-u)\cos 2u\,du$$

$$=\frac{1}{2}\left\{\int_0^t \cos 2t\,du+\int_0^t\cos(2t-4u)du\right\}=\frac{1}{2}\left\{t\cos 2t+\left[-\frac{1}{4}\sin(2t-4u)\right]_0^t\right\}$$

$= \frac{1}{2}\left\{t\cos 2t - \frac{1}{4}\left(\sin(-2t) - \sin 2t\right)\right\} = \frac{1}{2}\left(t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right)$

$= \frac{t}{2}\cos 2t + \frac{1}{4}\sin 2t$

$\boxed{4}$ (4) $\mathcal{L}(f(t)) = \tan^{-1}\frac{s}{2}$ とおくと $\mathcal{L}(-tf(t)) = \frac{d}{ds}\tan^{-1}\frac{s}{2} = \frac{1}{2}\cdot\frac{1}{1+\left(\frac{s}{2}\right)^2} = \frac{1}{2}\cdot\frac{4}{s^2+4}$

$= \frac{2}{s^2+4} = \mathcal{L}(\sin 2t)$ $\therefore \mathcal{L}^{-1}\left(\tan^{-1}\frac{s}{2}\right) = f(t) = -\frac{\sin 2t}{t}$

(2) $\mathcal{L}(f(t)) = \log\frac{2s^2+1}{s(s^2+1)}$ とおくと $\mathcal{L}(-tf(t)) = \frac{d}{ds}\log\frac{2s^2+1}{s(s^2+1)}$

$= \frac{d}{ds}\left\{\log(2s^2+1) - \log s - \log(s^2+1)\right\} = \frac{4s}{2s^2+1} - \frac{1}{s} - \frac{2s}{s^2+1}$

$= 2\cdot\frac{s}{s^2+\frac{1}{2}} - \mathcal{L}(1) - 2\mathcal{L}(\cos t) = 2\mathcal{L}\left(\cos\frac{t}{\sqrt{2}}\right) - \mathcal{L}(1) - 2\mathcal{L}(\cos t)$

$= \mathcal{L}\left(2\cos\frac{t}{\sqrt{2}} - 1 - 2\cos t\right)$ $\therefore \mathcal{L}^{-1}\left(\log\frac{2s^2+1}{s(s^2+1)}\right) = f(t) = \frac{1+2\cos t - 2\cos\frac{t}{\sqrt{2}}}{t}$

(3) $\mathcal{L}(f(t)) = \log\frac{s^2+1}{s^2-1}$ とおくと $\mathcal{L}(-tf(t)) = \frac{d}{ds}\log\frac{s^2+1}{s^2-1}$

$= \frac{d}{ds}\left\{\log(s^2+1) - \log(s^2-1)\right\} = \frac{2s}{s^2+1} - \frac{2s}{s^2-1} = 2\mathcal{L}(\cos t) - 2\mathcal{L}(\cosh t)$

$= \mathcal{L}\left(2(\cos t - \cosh t)\right)$ $\therefore \mathcal{L}^{-1}\left(\log\frac{s^2+1}{s^2-1}\right) = f(t) = \frac{2(\cosh t - \cos t)}{t}$

(1) $\mathcal{L}(f(t)) = \sin^{-1}\frac{s}{\sqrt{1+s^2}}$ とおくと $\mathcal{L}(-tf(t)) = \frac{d}{ds}\sin^{-1}\frac{s}{\sqrt{1+s^2}}$

$= \frac{1}{\sqrt{1-\left(\frac{s}{\sqrt{1+s^2}}\right)^2}}\cdot\frac{\sqrt{1+s^2} - s\cdot\frac{1}{2}\cdot\frac{2s}{\sqrt{1+s^2}}}{1+s^2} = \frac{\sqrt{1+s^2}}{\sqrt{1+\cancel{s^2}-\cancel{s^2}}}\cdot\frac{\sqrt{1+s^2} - \frac{s^2}{\sqrt{1+s^2}}}{1+s^2}$

$= \frac{1+s^2-s^2}{1+s^2} = \frac{1}{s^2+1} = \mathcal{L}(\sin t)$ $\therefore f(t) = -\frac{\sin t}{t}$ $\therefore \mathcal{L}^{-1}\left(\sin^{-1}\frac{s}{\sqrt{1+s^2}}\right) = -\frac{\sin t}{t}$