問題2.2 解答

1.(1) $L(t^2+3t+4)=L(t^2)+3L(t)+4L(1)$

$$=\frac{2}{s^3}+\frac{3}{s^2}+\frac{4}{s}$$

(2) $L((t-1)^3)=L(t^3-3t^2+3t-1)=\frac{6}{s^4}-\frac{6}{s^3}+\frac{3}{s^2}-\frac{1}{s}$

(3) $L(\cos(2t+\frac{\pi}{6}))=L(\frac{\sqrt{3}}{2}\cdot\cos 2t-\frac{1}{2}\sin 2t)$

$$=\frac{\sqrt{3}}{2}\cdot\frac{s}{s^2+4}-\frac{1}{2}\cdot\frac{2}{s^2+4}=\frac{\sqrt{3}}{2}\cdot\frac{s}{s^2+4}-\frac{1}{s^2+4}$$

(4) $L(\sinh t+\sin t)=\frac{1}{s^2-1}+\frac{1}{s^2+1}$

(5) $f(t)=(t+1)^2$ とすると. $L(f)=L(t^2+2t+1)=\frac{2+2s+s^2}{s^3}$ より

$L((t+1)^2e^{3t})=L(f)(s-3)$

$$=\frac{2+2(s-3)+(s-3)^2}{(s-3)^3}=\frac{s^2-4s+5}{(s-3)^2}$$

(6) $L(e^{-t}\cdot\cos t)=L(\cos t)(s+1)$

$$=\frac{s+1}{(s+1)^2+1}=\frac{s+1}{s^2+2s+2}$$

(7) $L(f(t))=L((\cos t)_2)=e^{-2s}\cdot\frac{s}{s^2+1}$

2.(1) $L(t\sin t)=-L(-t\sin t)=-\frac{d}{ds}L(\sin t)=-\frac{d}{ds}\frac{1}{s^2+1}=\frac{2s}{(s^2+1)^2}$

(2) $L(t\cos t)=L((-t)^2\cos t)=\frac{d^2}{ds^2}L(\cos t)=\frac{d^2}{ds^2}\frac{s}{s^2+1}$

$$=\frac{d}{ds}\cdot\frac{s^2+1-2s^2}{(s^2+1)^2}=\frac{d}{ds}\frac{-s^2+1}{(s^2+1)^2}=\frac{-2s(s^2+1)^2-4s(s^2+1)(-s^2+1)}{(s^2+1)^4}=\frac{2s(s^2-3)}{(s^2+1)^3}$$

(3) $L(t\cosh 2t) = -\dfrac{d}{ds}\cdot\dfrac{s}{s^2-4} = -\dfrac{s^2-4-2s^2}{(s^2-4)^2} = \dfrac{s^2+4}{(s^2-4)^2}$

(4) $L(t^2\sinh t) = \dfrac{d^2}{ds^2}\,\dfrac{1}{s^2-1} = \dfrac{d}{ds}\,\dfrac{-2s}{(s^2-1)^2} = \dfrac{-2(s^2-1)^2+8s^2(s^2-1)}{(s^2-1)^4}$

$= \dfrac{6s^2+2}{(s^2-1)^3}$

3. (1) $L\left(\dfrac{1-e^{2t}}{t}\right) = \int_s^\infty L(1-e^{2t})(p)\,dp = \int_s^\infty \dfrac{1}{p} - \dfrac{1}{p-2}\,dp$

$= \left[\log\dfrac{p}{p-2}\right]_s^\infty = -\log\dfrac{s}{s-2} = \log\dfrac{s-2}{s}$

(2) $L\left(\dfrac{e^t-\cos t}{t}\right) = \int_s^\infty \dfrac{1}{p-1} - \dfrac{p}{p^2+1}\,dp = \left[\log(p-1) - \dfrac{1}{2}\log(p^2+1)\right]_s^\infty$

$= \left[\log\dfrac{p-1}{(p^2+1)^{\frac{1}{2}}}\right]_s^\infty = -\log\dfrac{s-1}{(s^2+1)^{\frac{1}{2}}} = \log\dfrac{(s^2+1)^{\frac{1}{2}}}{s-1}$

(3) $L\left(\dfrac{\cos t-\cos 2t}{t}\right) = \int_s^\infty \dfrac{p}{p^2+1} - \dfrac{p}{p^2+4}\,dp = \left[\dfrac{1}{2}\log(p^2+1) - \dfrac{1}{2}\log(p^2+4)\right]_s^\infty$

$= \dfrac{1}{2}\left(-\log(s^2+1) + \log(s^2+4)\right)$

(4) $L\left(\dfrac{\sinh t}{t}\right) = \int_s^\infty \dfrac{1}{p^2-1}\,dp = \dfrac{1}{2}\int_s^\infty \dfrac{1}{p-1} - \dfrac{1}{p+1}\,dp$

$= \dfrac{1}{2}\left[\log(p-1) - \log(p+1)\right]_s^\infty = \dfrac{1}{2}\left(\log(s+1) - \log(s-1)\right)$

4. (1) $L(\sin^2 t) = L\left(\dfrac{1-\cos 2t}{2}\right) = \dfrac{1}{2s} - \dfrac{1}{2}\cdot\dfrac{s}{s^2+4} = \dfrac{2}{s(s^2+4)}$

(2) $L(\sin^3 t) = L\left(-\dfrac{1}{4}\sin 3t + \dfrac{3}{4}\sin t\right) = \dfrac{1}{4}\left(-\dfrac{3}{s^2+9} + 3\cdot\dfrac{1}{s^2+1}\right)$

$= \dfrac{3}{4}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9}\right) = \dfrac{6}{(s^2+1)(s^2+9)}$

(3) $L(\sin t \cos 2t) = L\left(\frac{1}{2}(\sin 3t - \sin t)\right) = \frac{1}{2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right)$

$= \frac{s^2-3}{(s^2+1)(s^2+9)}$

(4) $L(\cos t \cos 2t) = L\left(\frac{1}{2}(\cos 3t + \cos t)\right) = \frac{1}{2}\left(\frac{s}{s^2+9} + \frac{s}{s^2+1}\right)$

$= \frac{s(s^2+5)}{(s^2+1)(s^2+9)}$

5.(1) $L\left(\int_0^t e^{3u}du\right) = \frac{1}{s}L(e^{3u}) = \frac{1}{s}\cdot\frac{1}{s-3}$

$L\left(\int_0^t e^{3u}du\right) = L\left(\left[\frac{1}{3}e^{3u}\right]_0^t\right) = L\left(\frac{1}{3}(e^{3t}-1)\right)$

$= \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right) = \frac{1}{s(s-3)}$

(2) $L\left(\int_0^t \sin 2u\, du\right) = \frac{1}{s}L(\sin 2u) = \frac{1}{s}\cdot\frac{2}{s^2+4}$

$L\left(\int_0^t \sin 2u\, du\right) = L\left(\left[-\frac{1}{2}\cos 2u\right]_0^t\right) = L\left(\frac{1}{2}(-\cos 2t + 1)\right)$

$= \frac{1}{2}\left(\frac{-s}{s^2+4} + \frac{1}{s}\right) = \frac{2}{s(s^2+4)}$

(3) $L\left(\int_0^t\int_0^u \cos 2v\, dv\, du\right) = \frac{1}{s^2}L(\cos 2v) = \frac{1}{s^2}\cdot\frac{s}{s^2+4} = \frac{1}{s(s^2+4)}$

$L\left(\int_0^t\int_0^u \cos 2v\, dv\, du\right) = L\left(\int_0^t \left[\frac{1}{2}\sin 2v\right]_0^u du\right) = L\left(\int_0^t \frac{1}{2}\sin 2u\, du\right)$

$= \frac{1}{2}L\left(\left[-\frac{1}{2}\cos 2u\right]_0^t\right) = -\frac{1}{4}L(\cos 2t - 1) = -\frac{1}{4}\left(\frac{s}{s^2+4} - \frac{1}{s}\right) = \frac{1}{s(s^2+4)}$

6.(1) $L(t^2 * e^{2t}) = \frac{2}{s^3} \cdot \frac{1}{s-2}$

$$t^2 * e^{2t} = \int_0^t u^2 \cdot e^{2(t-u)} du = \left[-\frac{1}{2}u^2 e^{2(t-u)}\right]_0^t + \int_0^t u e^{2(t-u)} du$$

$$= -\frac{1}{2}t^2 + \left[-\frac{1}{2}u e^{2(t-u)}\right]_0^t + \frac{1}{2}\int_0^t e^{2(t-u)} du$$

$$= -\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{2}\left[-\frac{1}{2}e^{2(t-u)}\right]_0^t = -\frac{1}{4}(2t^2+2t+1-e^{2t})$$ より

$$L(t^2 * e^{2t}) = -\frac{1}{4}\left(\frac{4}{s^3} + \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s-2}\right) = \frac{2}{s^3(s-2)}$$

(2) $L(\sin t * \cos t) = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$

$$\sin t * \cos t = \int_0^t \sin u \cdot \cos(t-u) du = \frac{1}{2}\int_0^t \sin t - \sin(2u-t) du$$

$$= \frac{1}{2}\left[u \sin t + \frac{1}{2}\cos(2u-t)\right]_0^t = \frac{1}{2}\left(t \sin t + \frac{1}{2}\cos t - \frac{1}{2}\cos(-t)\right) = \frac{1}{2}t \sin t$$

2.(1) より $L(\sin t * \cos t) = L\left(\frac{1}{2}t \sin t\right) = \frac{s}{(s^2+1)^2}$.

(3) $L(e^{2t} * \sin t) = \frac{1}{s-2} \cdot \frac{1}{s^2+1}$

$$e^{2t} * \sin t = \int_0^t e^{2u} \sin(t-u) du = \left[e^{2u}\cos(t-u)\right]_0^t - 2\int_0^t e^{2u}\cos(t-u) du$$

$$= e^{2t} - \cos t - 2\left[-e^{2u}\sin(t-u)\right]_0^t - 4\int_0^t e^{2u}\sin(t-u) du$$ より.

$$e^{2t} * \sin t = \frac{1}{5}(e^{2t} - \cos t - 2\sin t)$$

$$\therefore L(e^{2t} * \sin t) = \frac{1}{5}\left(\frac{1}{s-2} - \frac{s}{s^2+1} - \frac{2}{s^2+1}\right) = \frac{1}{(s-2)(s^2+1)}$$