問題1.3.3

1 推測特殊解 y_0 は次の通り.

(1) $y_0 = a_0x^2+a_1x+a_2$ (2) $y_0 = ae^{x}$ (3) $y_0=(a_0x+a_1)e^{-x}$

(4) $y_0 = a\sin x + b\cos x$ (5) $y_0 = ae^{x}\sin x + be^{x}\cos x$

(6) $2e^{3x}\to ae^{3x}$, $\sin x \to b\sin x + c\cos x$ ∴ $y_0 = ae^{3x}+b\sin x + c\cos x$

(7) $6x\to a_0x+a_1$, $8e^{2x}$ は基本解 $y=e^{2x}$ の定数倍なので $8e^{2x}\to bxe^{2x}$

∴ $y_0 = a_0x+a_1+bxe^{2x}$

(8) $y_0=a_0x^2+a_1x+a_2$ (9) $y_0=(a_0x+a_1)e^{x}\sin 2x+(b_0x+b_1)e^{x}\cos 2x$

(10) $\cos 2x \to a\sin 2x + b\cos 2x$, $5x\to cx+d$ ∴ $y_0 = a\cos 2x + b\cos 2x + cx+d$

(11) 基本解が定数関数を含むので. $2x\to x(a_0x+a_1)$, $3\cos x\to b_1\sin x + b_2\cos x$,

e^{-x} は基本解の定数倍なので. $e^{-x}\to cxe^{-x}$ 以上より.

$y_0 = x(a_0x+a_1)+(b_1\sin x+b_2\cos x)+cxe^{-x}$

2 (i) $\lambda^2+6\lambda+8=(\lambda+2)(\lambda+4)=0$ ∴ $\lambda=-2,-4$. よって基本解は $y_1=e^{-2x}$, $y_2=e^{-4x}$.

よって. $W(y_1,y_2) = \begin{vmatrix} e^{-2x} & e^{-4x}\\ -2e^{-2x} & -4e^{-4x}\end{vmatrix} = -4e^{-6x}+2e^{-6x}=-2e^{-6x}$

$$\therefore \frac{-R(x)y_2}{W(y_1,y_2)} = \frac{-\frac{2}{1+e^{2x}}\cdot e^{-4x}}{-2e^{-6x}} = \frac{e^{-4x}}{1+e^{2x}}\cdot e^{6x} = \frac{e^{2x}}{1+e^{2x}}$$

$$\therefore \int \frac{-R(x)y_2}{W(y_1,y_2)}dx = \int \frac{e^{2x}}{1+e^{2x}}dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}dx = \frac{1}{2}\log(1+e^{2x})$$

一方 $\dfrac{R(x)y_1}{W(y_1,y_2)} = \dfrac{\frac{2}{1+e^{2x}}\cdot e^{-2x}}{-2e^{-6x}} = -\dfrac{e^{-2x}}{1+e^{2x}}\cdot e^{6x} = -\dfrac{e^{4x}}{1+e^{2x}} = -\dfrac{e^{2x}(1+e^{2x})-e^{2x}}{1+e^{2x}}$

$= \dfrac{e^{2x}}{1+e^{2x}} - e^{2x}$

$\therefore \displaystyle\int \frac{R(x)y_1}{W(y_1,y_2)}dx = \int\left(\frac{e^{2x}}{1+e^{2x}} - e^{2x}\right)dx = \frac{1}{2}\log(1+e^{2x}) - \frac{1}{2}e^{2x}$

$\therefore y_0 = e^{-2x}\cdot\frac{1}{2}\log(1+e^{2x}) + e^{-4x}\left\{\frac{1}{2}\log(1+e^{2x}) - \frac{1}{2}e^{2x}\right\}$

$= -\frac{1}{2}e^{-2x} + \frac{1}{2}(e^{-2x}+e^{-4x})\log(1+e^{2x})$

$\therefore y = C_1e^{-2x} + C_2e^{-4x} - \frac{1}{2}e^{-2x} + \frac{1}{2}(e^{-2x}+e^{-4x})\log(1+e^{2x})$

$\therefore y_1 = C_1e^{-2x} + C_2e^{-4x} + \frac{1}{2}(e^{-2x}+e^{-4x})\log(1+e^{2x}) \quad \left(C_1 - \frac{1}{2} \to C_1\right)$

(2) $\lambda^2+1=0 \quad \therefore \lambda = \pm i$ より基本解は $y_1=\sin x$, $y_2=\cos x$ ゆえに

$W(y_1,y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$

$\therefore \dfrac{-R(x)y_2}{W(y_1,y_2)} = \dfrac{-\frac{1}{\cos x}\cdot\cos x}{-1} = 1 \quad \therefore \displaystyle\int\frac{-R(x)y_2}{W(y_1,y_2)}dx = \int 1\,dx = x$

一方,

$\dfrac{R(x)y_1}{W(y_1,y_2)} = \dfrac{\frac{1}{\cos x}\sin x}{-1} = \dfrac{-\sin x}{\cos x} \quad \therefore \displaystyle\int\frac{R(x)y_1}{W(y_1,y_2)}dx = \int\frac{-\sin x}{\cos x}dx = \log|\cos x|$

$\therefore y_0 = x\sin x + (\cos x)\log|\cos x| \quad \therefore y = C_1\sin x + C_2\cos x + x\sin x + (\cos x)\log|\cos x|$