

STRONG AND WEAK (1, 2, 3) HOMOTOPIES ON KNOT PROJECTIONS

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ABSTRACT. A knot projection is an image of a generic immersion from a circle into a two-dimensional sphere. We can find homotopies between any two knot projections by local replacements of knot projections of three types, called Reidemeister moves. This paper defines an equivalence relation for knot projections called weak (1, 2, 3) homotopy, which consists of Reidemeister moves of type 1, weak type 2, and weak type 3. This paper defines the first non-trivial invariant under weak (1, 2, 3) homotopy. We use this invariant to show that there exist an infinite number of weak (1, 2, 3) homotopy equivalence classes of knot projections. By contrast, all equivalence classes of knot projections consisting of the other variants of a triple type, i.e., Reidemeister moves of (1, strong type 2, strong type 3), (1, weak type 2, strong type 3), and (1, strong type 2, weak type 3), are contractible.

1. INTRODUCTION.

A *knot projection* is defined as an image of a generic immersion of a circle into a two-dimensional sphere. A *knot diagram* is a knot projection specifying information of over/under-crossing branches. A *trivial* knot projection is defined as a knot projection with no double points, namely a simple closed curve on a 2-sphere. It is well known that an arbitrary knot projection can be related to a trivial knot projection by through RI (Reidemeister move of type 1), RII (Reidemeister move of type 2), and RIII (Reidemeister move of type 3), as shown in Fig. 1. Here, each crossing in Fig. 1 is a *flat crossing*. A flat crossing is a double point of a knot diagram without over/under information.



FIGURE 1. RI, RII, and RIII.

RII (resp. RIII) can be decomposed into exactly two types, namely, strong RII (resp. strong RIII) and weak RII (resp. weak RIII), as shown in Fig. 2. Here, we introduce equivalence relations, called *strong* (1, 2, 3) *homotopy* and *weak* (1, 2,

Date: June 6, 2015.

Key words and phrases. generic spherical curves; knot projections; Reidemeister moves; homotopy.

MSC2010, MSC2000: 57M25; 57Q35.

The work of N. Ito was partly supported by a Waseda University Grant for Special Research Projects (Project number: 2014K-6292) and the JSPS Japanese-German Graduate Externship.

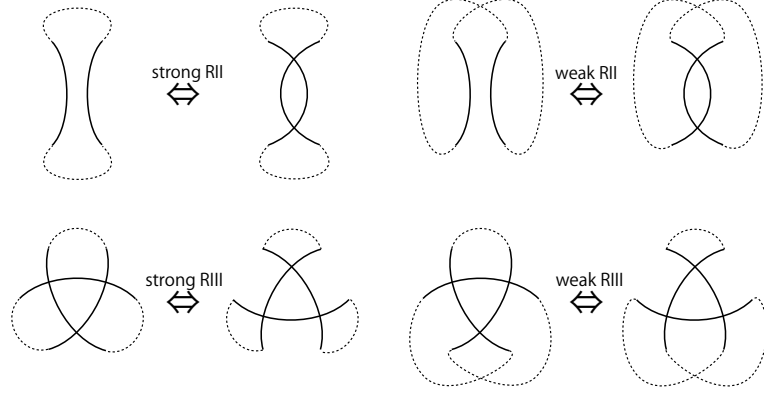


FIGURE 2. Strong RII and weak RII (upper), strong RIII, and weak RIII (lower).

3) *homotopy*, which are defined as follows. Two knot projections are said to be *strongly* (resp. *weakly*) (1, 2, 3) *homotopic* if the two knot projections are related by a finite sequence consisting of RI, strong RII (resp. weak RII), and strong RIII (resp. weak RIII).

Theorem 1. (1) Any two knot projections are strongly (1, 2, 3) homotopic.
 (2) There exists infinitely many knot projections $\{P_i\}$ that are not weakly (1, 2, 3) homotopic to a trivial knot projection, where P_i is not weakly (1, 2, 3) homotopic to P_j for any two nonnegative integers i and j , such that $i \neq j$.

To prove Theorem 1 (2), we find a non-negative integer-valued new invariant W under RI, weak RII, and weak RIII.

Here, we explain the historical position of this study. Equivalence classes of knot projections, on a plane and sphere, have been studied by many researchers. For example, Arnold introduced invariants, J^+ , J^- , and St , for plane curves [1]. However, almost all these studies treat plane curves (i.e., knot projections on a plane) under regular homotopy, i.e., without RI. To the best of our knowledge, only $J^+ + 2St$, given by the Arnold's invariants, is invariant under RI for knot projections on a sphere. Studies of equivalence classes of knot projections under homotopy containing RI are: [10, 6] for RI and RII, [8, 7] for a pair of RI and strong or weak RII, [2] for RI and RIII, [9] for RI and strong RIII, and [6] for RI and weak RIII.

However, this is the first paper that considers equivalence classes consisting of a triple of Reidemeister moves: (RI, strong RII, strong RIII) and (RI, weak RII, weak RIII).

The remainder of this paper is organized as follows. Section 2 defines a new invariant W under RI, weak RII, and weak RIII. Section 3 introduces properties of W . Section 4 obtains a proof of Theorem 1. Section 5 demonstrates that every knot projection under each equivalence relation consisting of the other variants of a triple tuple of Reidemeister moves, namely (RI, strong RII, weak RIII) or (RI, weak RII, strong RIII), is equivalent to a trivial knot projection.

2. DEFINITION OF A NEW INVARIANT W .

Let P be an arbitrary knot projection and the number of double points be denoted by $c(P)$. *Seifert circle number* $s(P)$ is defined as follows. When P has

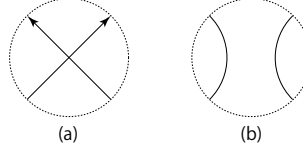


FIGURE 3. (a) Disk as a neighborhood of a double point; (b) Disk corresponding to (a) after applying the replacement.

an arbitrary orientation σ , a sufficient small neighborhood of each double point is isotopic to Fig. 3 (a). We replace the disk in Fig. 3 (a) with that in Fig. 3 (b). After replacing all the double points, we obtain an arrangement $S(P)$ of circles on the sphere. By definition, $S(P)$ does not depend on the orientation σ of P . Each circle in $S(P)$ is called a *Seifert circle*. Seifert circle number $s(P)$ is defined as the number of circles in $S(P)$. It is well-known that for any knot diagram D_P obtained by arbitrarily obtaining information of any over/under-crossing branches, the *canonical genus* of D_P is equivalent to $c(P) - s(P) + 1$; $c(P) - s(P) + 1$ is denoted by $g(P)$. Note that all knot diagrams obtained by the knot projection P have the same canonical genus.

A *chord diagram* of the knot projection P is defined as the immersing circle with even points on the circle, which correspond to the preimages of double points of P where each pair of preimages of a double point is connected by a chord. The *trivializing number* $tr(P)$ of the knot projection P is the minimum number of erased chords in the chord diagram of P until there is no cross chord, i.e., there are no intersecting chords as $\otimes$, embedded in the whole chord diagram [4, Page 440, Theorem 13].

Theorem 2. *Let P be an arbitrary knot projection. Then,*

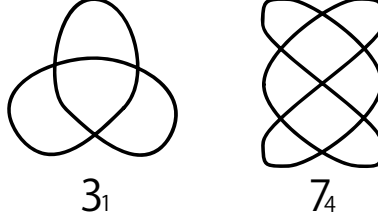
$$W(P) = tr(P) - 2g(P)$$

is invariant under RI, weak RII, and weak RIII.

Proof. We check the differences $tr(P)$, $s(P)$, and $c(P)$ because $W(P) = tr(P) - 2g(P) = tr(P) + s(P) - c(P) - 1$. By definitions, it is clear that $c(P)$ and $s(P)$ are invariant under weak RIII. We also recall that there is a fact that $tr(P)$ is invariant under RI and weak RIII (cf. [9]). Thus, we check the differences before and after the application RI and weak RII (Table 1). A single RI (resp. weak RII) with increasing $c(P)$ is denoted by $1a$ (resp. $w2a$). $\square$

TABLE 1. The differences before and after the application of $1a$ and $w2a$.

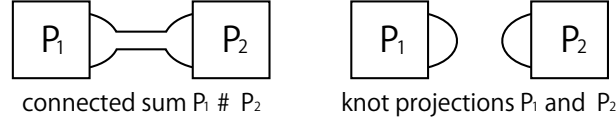
	$tr(P)$	$s(P)$	$c(P)$
$1a$	0	+1	+1
$w2a$	+2	0	+2

FIGURE 4. 3_1 and 7_4 .

Example 1. Let 3_1 and 7_4 be knot projections, as shown in Fig. 4. $W(3_1) = tr(3_1) + s(3_1) - c(3_1) - 1 = 2 + 2 - 3 - 1 = 0$. $W(7_4) = tr(7_4) + s(7_4) - c(7_4) - 1 = 4 + 6 - 7 - 1 = 2$.

3. PROPERTIES OF W .

In this section, we introduce the properties of W . The *connected sum* of P_1 and P_2 is defined in Fig. 5. If a knot projection P is non-trivial and cannot be a

FIGURE 5. Connected sum $P_1 \sharp P_2$ for knot projections P_1 and P_2 .

connected sum of non-trivial knot projections, P is called a *prime* knot projection.

Theorem 3. *Let P and P' be arbitrary two knot projections. Then, $W(P)$ has the following properties.*

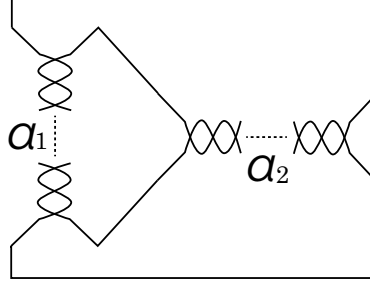
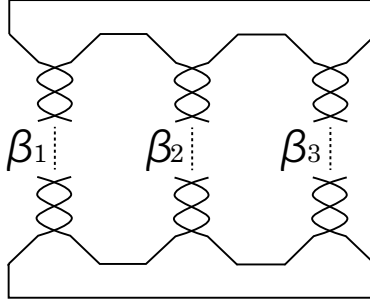
- (1) $W(P)$ is an even integer.
- (2) $W(P \sharp P') = W(P) + W(P')$.
- (3) For an arbitrary even nonnegative integer m , there exists a knot projection P such that $W(P) = m$. Further, a prime knot projection P can be chosen.
- (4) $0 \leq W(P) \leq c(P) - 1$ for an arbitrary non-trivial knot projection P .
- (5) If $W(P) = c(P) - 1$, P is the knot projection that appears as ∞ .

Proof. (1) $tr(P)$, $s(P)$, and $c(P)$ change by ± 2 or 0 under a single strong RII or a single strong RIII. Thus, we have the claim.

- (2) It is clear that $tr(P \sharp P') = tr(P) + tr(P')$, $s(P \sharp P') = s(P) + s(P') - 1$, and $c(P \sharp P') = c(P) + c(P')$.

- (3) Note that $W(3_1) = 0$ and $W(7_4) = 2$ (see Example 1). Thus, by using Theorem 3 (2) and considering any connected sum, we have the former claim. For the latter claim, consider a rational knot projection $p(\alpha_1, \alpha_2)$ with a pair of even integers (α_1, α_2) such that $\alpha_1 \geq \alpha_2 \geq 4$, as shown in Fig. 6, where α_1 and α_2 represent the number of double points. We have $tr(p(\alpha_1, \alpha_2)) = \alpha_2$, $s(p(\alpha_1, \alpha_2)) = \alpha_1 + \alpha_2 - 1$, and $c(p(\alpha_1, \alpha_2)) = \alpha_1 + \alpha_2$. Thus, $W(p(\alpha_1, \alpha_2)) = \alpha_2 - 2$.

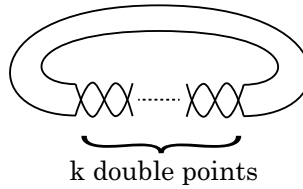
Alternatively, consider a pretzel knot projection $q(\beta_1, \beta_2, \beta_3)$ with a tuple of odd integers $(\beta_1, \beta_2, \beta_3)$ such that $\beta_1 \geq \beta_2 \geq \beta_3 \geq 3$, as shown in Fig. 7, where β_1 , β_2 , and β_3 represent the number of double points. We have $tr(q(\beta_1, \beta_2, \beta_3)) = \beta_2 + \beta_3$, $s(q(\beta_1, \beta_2, \beta_3)) = \beta_1 + \beta_2 + \beta_3 - 1$, and $c(q(\beta_1, \beta_2, \beta_3)) = \beta_1 + \beta_2 + \beta_3$. Thus, $W(q(\beta_1, \beta_2, \beta_3)) = \beta_2 + \beta_3 - 2$.

FIGURE 6. $p(\alpha_1, \alpha_2)$.FIGURE 7. $q(\beta_1, \beta_2, \beta_3)$.

- (4) $0 \leq tr(P) - 2g(P)$ ($= W(P)$) is obtained by a known inequality (see [5, Proof of Theorem 7.11]). We can see that $s(P) \leq c(P) + 1$ and $tr(P) \leq c(P) - 1$ if $1 \leq c(P)$. Thus,

$$W(P) = tr(P) + s(P) - c(P) - 1 \leq c(P) - 1.$$

- (5) If $W(P) = c(P) - 1$, $tr(P) = c(P) - 1$, and $s(P) = c(P) + 1$. From [3, Theorem 1.11], if $tr(P) = c(P) - 1$, P is a knot projection as shown in Fig. 6. In this case, $c(P) + 1 = s(P) = 2$. Then, $c(P) = 1$.

FIGURE 8. A series of knot projections where k is a positive odd integer.

□

4. PROOF OF THEOREM 1.

Proof. Proof of (1). It can be seen that a single weak RII and a single weak RIII are generated by a finite sequence consisting of RI, strong RII, and strong RIII from Fig. 9.

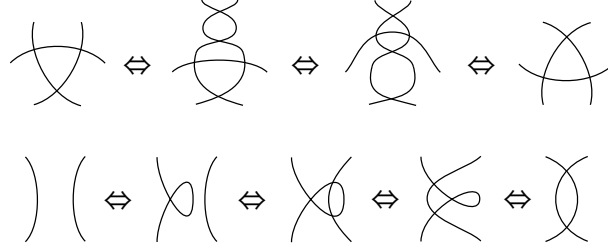


FIGURE 9. A single RIII consisting of two RIIs and a single RIII (upper line). A single RII consisting of two RIs, a single RII, and a single RIII (lower line).

Proof of (2). Let O be a trivial knot projection, i.e., a simple closed curve. Here, $W(O) = 0$ and $W(7_4) = 2$. Since $W(P)$ holds the property of Theorem 3 (3), we have the claim. $\square$

5. OTHER EQUIVALENCES.

Definition 1. (1) Two knot projections P and P' are $(1, s2, w3)$ homotopic if P and P' are related by a finite sequence consisting of RI, strong RII, and weak RIII. (2) Two knot projections P and P' are $(1, w2, s3)$ homotopic if P and P' are related by a finite sequence consisting of RI, weak RII, and strong RIII.

Proposition 1. (1) Any two knot projections are $(1, s2, w3)$ homotopic. (2) Any two knot projections are $(1, w2, s3)$ homotopic.

Proof. Recall Theorem 1 (1): an arbitrary knot projection can be related to a trivial knot projection O by a finite sequence consisting of RI, strong RII, and strong RIII. Thus, we verify that a single strong RIII (resp. strong RII) is created in the case (1) (resp. (2)) as follows.

- (1) From Fig. 9 (upper line), we can see that a single strong RIII consists of two strong RII and a single weak RIII.
- (2) From Fig. 9 (lower line), we can see that a single strong RII consists of two RIs, a single weak RII and a single strong RIII. $\square$

ACKNOWLEDGEMENTS.

The authors would like to thank Professor Kouki Taniyama for his variable comments.

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