A New Aspect of the Arnold Invariant J^+ from a Global Viewpoint

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ABSTRACT. In this paper, we study the Arnold invariant J^+ for plane and spherical curves. This invariant essentially counts the number of a certain type of local moves called *direct self-tangency perestroika* in a generic regular homotopy from a standard curve to a given one; the other basic local moves, namely *inverse self-tangency perestroika* and *triple point crossing*, do not change the value of J^+. Thus, behavior of J^+ under local moves is rather obvious. However, it is less understood how J^+ behaves in the space of curves on a global scale. We study this problem using Legendrian knots, and give infinitely many regular homotopic curves with the same J^+ that cannot be mutually related by inverse self-tangency perestroika and triple point crossing.

1. INTRODUCTION

In this paper, we study the images of generic immersions (i.e., immersions such that all of the self-intersections are transverse double points) from the circle S^1 to the plane $\mathbb{R}^2$ and the sphere S^2, which we call *plane curves* and *spherical curves*, respectively. Two curves are said to be *regular homotopic* if these are connected by a homotopy that is an immersion all the time. Whitney [14] proved that two plane curves are regular homotopic if and only if these curves have the same rotation number. It is known that any regular homotopy can be changed (by a small perturbation) into a finite sequence of three types of basic homotopies: called a *dR2-move*, an *iR2-move*, and an *R3-move* (cf. Figure 2.2; we give a quick review on these moves in Section 2.)

Arnold [3] introduced numerical invariants J^+, J^-, and St for plane curves by carefully analyzing the space of immersions from S^1 to $\mathbb{R}^2$, and further modified

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these invariants in order to obtain invariants J_S^+, J_S^- and St_S for spherical curves (see [5]). The Arnold invariants attracted much interest to a new aspect of curves, and have been studied a lot for the last two decades: several explicit formulae of the invariants were given in [10, 11, 13], higher-order versions were obtained in [2, 6, 13], generalizations to fronts were studied in [1, 4, 10], and so on.

As we will see in Section 2, J^+ (and thus the modified version J_S^+) is invariant under ambient isotopies, iR2-moves and R3-moves, but the values of J^+ differ by 2 between two curves if these curves are related by a dR2-move. Thus, for two curves C_1 and C_2, the difference $J^+(C_1) - J^+(C_2)$ gives a lower bound of the number of dR2-moves in a generic regular homotopy between C_1 and C_2. The invariants J^- and St (and the modified versions J_S^- and St_S) have similar properties (see Section 2). In particular, the Arnold invariants might be "obstructions" to the existence of a generic regular homotopy consisting of specific types of moves for a given two curves (cf. [3, p. 34]).

In order to further study the "obstructions" above, we introduce some rather unusual equivalence relations of curves: for a set R of types of moves for curves, two curves are said to be *equivalent under* R if one curve can be obtained from the other by successive applications of moves in R. Several equivalence relations of curves defined by restricted homotopies were studied in [7–9]. In this paper, we especially focus on the equivalence relation under $\{\text{iR2}, \text{R3}\}$. This equivalence relation is appropriate for Legendrian theory: dR2-moves are not allowed in Legendrian regular homotopies between fronts. The rotation number of an oriented plane curve is invariant under iR2-moves and R3-moves. For an oriented plane curve C, we denote the rotation number of C by $\text{rot}(C) \in \mathbb{Z}$. By the property in the previous paragraph, J^+ is also invariant under iR2-moves and R3-moves. We will prove the following theorem.

Theorem 1.1. *For any $r \in \mathbb{Z}$ and $j^+ \in 2\mathbb{Z}$, there exists an infinite family of plane curves $\{C_i\}$ that satisfies the following properties:*

- $\text{rot}(C_i) = r$ *and* $J^+(C_i) = j^+$ *for any* i.
- *For any i, j with $i \neq j$, C_i and C_j are not equivalent under* $\{\text{iR2}, \text{R3}\}$.

For spherical curves, we can define the rotation number modulo 2, which is invariant under regular homotopies, in particular under iR2-moves and R3-moves. We denote the rotation number of a spherical curve C by $\text{rot}_S(C) \in \mathbb{Z}/2\mathbb{Z}$. By the definition of J_S^+ (see Section 2), for a spherical curve C the value $J_S^+(C)$ is contained in $V_0 = \mathbb{Z}$ if the rotation number $\text{rot}_S(C)$ is equal to 0, and is contained in $V_1 = (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$ otherwise.

Theorem 1.2. *For any $r \in \mathbb{Z}/2\mathbb{Z}$ and any $j^+ \in V_r$, there exists an infinite family of spherical curves $\{C_i\}$ which satisfies the following properties:*

- $\text{rot}_S(C_i) = r$ *and* $J_S^+(C_i) = j^+$ *for any* i.
- *For any i, j with $i \neq j$, C_i and C_j are not equivalent under* $\{\text{iR2}, \text{R3}\}$.

The proofs of Theorems 1.1 and 1.2 will be given in Section 5. In order to detect difference between two curves up to equivalence under $\{\text{iR2}, \text{R3}\}$, we pay attention to isotopy classes of Legendrian knots in the unit tangent bundle UTS^2 of S^2 (i.e., the set of the tangent vectors of length 1 for some Riemannian metric of S^2) associated with oriented spherical curves. In the proofs of Theorems 1.1 and 1.2, we will give infinite families of oriented curves such that all the curves in a family have the same rotation number and the Arnold invariant, but where the associated Legendrian knots are mutually not isotopic. (This strategy for proving the main theorem should be compared with that in [8], in which the authors associated equivalence classes of curves under R1-moves and weak R3-moves with knots in S^3 by *positive resolution*.) Note that we never make use of contact structures of the unit tangent bundle for proving the main theorems: we merely study isotopy classes of Legendrian knots in the proof. Since UTS^2 is diffeomorphic to the real projective space $\mathbb{RP}^3$, UTS^2 has the surgery diagram consisting of a $(+2)$-framed unknot. In Section 3, we explain how to obtain a knot diagram of the Legendrian knot associated with a given curve, which is drawn in the surgery diagram of UTS^2. The unit tangent bundle UTS^2 has the universal double cover from S^3. By the covering homotopy property, if two links in UTS^2 are isotopic, the preimages of them under the universal cover are also isotopic. In Section 4, we give an algorithm to obtain a link diagram of the preimage of a link in UTS^2 under the universal cover, which enables us to detect the difference between two Legendrian knots up to isotopy.

2. PRELIMINARIES

In this section, we set up basic notation and terminology that will be used in this paper. Throughout the paper, we assume that manifolds are oriented and connected, and maps between manifolds are smooth unless otherwise noted.

2.1. Framed links and surgery diagrams of 3-manifolds. By a *link* we mean the image of an embedding from a disjoint union of oriented circles into a closed 3-manifold. A link L is called a *knot* if L is connected. For a knot K in M, we denote the closure of a tubular neighborhood of K by $\nu K \subset M$. The isotopy class of a circle in $\partial \nu K$ intersecting the meridian of K in one point is called a *framing*, and a knot with a framing is called a *framed knot*. A framing of a link L is a disjoint union of framings of each component of L. We call a link with a framing a *framed link*. A knot $K \subset S^3$ has the canonical framing: the framing K' such that the linking number between K and K' is 0. We denote this framing by ℓ, and the positive meridian of K by m. Since any framing of K is uniquely determined by its homology class in $\partial \nu K$, we can describe a framing of K by an integer: the framing corresponding to the homology class $p[m] + [\ell]$ is denoted by $p \in \mathbb{Z}$, which is called a *framing coefficient*.

We identify the sphere S^3 with the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$. We are interested in properties of links invariant under ambient isotopies of S^3. Since two links in S^3 are isotopic in S^3 if and only if these are isotopic in $S^3 \setminus \{\infty\}$, we can assume that any link in S^3 is away from ∞ without loss of generality. We can further assume that any link is in general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$; that is, all the self-intersections of the image of a link under p are transverse double points. For a knot $K \subset S^3$, we call the image of K under p a *knot projection* of K. We can add information of overpasses and underpasses to each double point of a knot projection. A knot projection with such information is called a *knot diagram*. Examples of a knot projection and knot diagrams are shown in Figure 2.1.

(a) (b) (c)

FIGURE 2.1. Above are a knot projection (a), and knot diagrams (b) and (c). The knot described in (b) is isotopic to that described in (c).

Let M be a 3-manifold. A framed knot $K \subset M$ gives rise to a new 3-manifold M_K in the following way: we take a diffeomorphism $\Phi : S^1 \times \partial D^2 \to \partial \nu K$ such that Φ maps the circle $\{*\} \times \partial D^2$ to the given framing of K (up to isotopy). Define a 3-manifold M_K as follows:

$$M_K = (M \setminus \mathrm{Int}(\nu K)) \cup_\Phi S^1 \times D^2.$$

It is not hard to see that the diffeomorphism type of M_K does not depend on the choice of Φ. The manifold M_K is called a *3-manifold obtained by Dehn surgery along* K. We can define Dehn surgery along a framed link $L \subset M$ in a similar manner. It is known that any closed 3-manifold can be obtained by Dehn surgery along a framed link $L \subset S^3$. For a 3-manifold M, a link diagram (with framing coefficients) of a framed link L that satisfies $M_L = M$ is called a *surgery diagram* of M.

Let M be a 3-manifold and L_0 a framed link in S^3 that satisfies $M_{L_0} = M$. We can regard the complement $S^3 \setminus \mathrm{Int}(\nu L_0)$ as a subset of M. It is easy to see that any link L in M can be moved by an isotopy so that L is contained in $S^3 \setminus \mathrm{Int}(\nu L_0)$. In particular, we can draw link diagrams L and L_0 simultaneously. Such a diagram is called a *link diagram* of L in a surgery diagram of M.

2.2. Regular homotopies of curves and the Arnold invariants. In this subsection, we give a quick review for generic homotopies between curves and the Arnold invariants. The reader can refer to [3, 4] for details on this subject.

For a curve C, we introduce three types of local moves, which are shown in Figure 2.2.

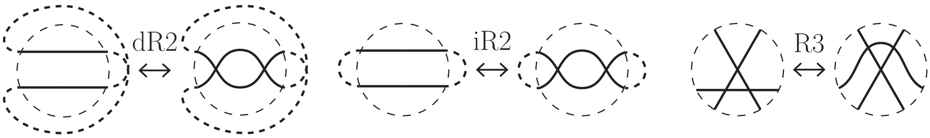

FIGURE 2.2. Three moves of curves.

The left move in Figure 2.2 can be realized by a regular homotopy which experiences a self-tangency at an intermediate time. In this homotopy, the two direction vectors have the same orientation at the self-tangency. This move is called a *direct self-tangency perestroika* or simply a *dR2-move*. The middle move in Figure 2.2 can be also realized by a regular homotopy with a self-tangency, but the orientations of the two direction vectors do not coincide at the self-tangency. This move is called an *inverse self-tangency perestroika* or an *iR2-move*. A dR2-move or an iR2-move is said to be *positive* (respectively, *negative*) if the move increases (respectively, decreases) the number of double points by 2.

The right move in Figure 2.2 is called a *triple point crossing* or an *R3-move*. This move can be realized by a regular homotopy that experiences a triple point. Although we can define *positive* and *negative* R3-moves, we will not discuss them in this paper. For this reason, we omit the details of positivity and negativity for R3-moves (the reader can refer to [3]).

It is known that any regular homotopy can be changed (by a small perturbation) into a finite sequence of the three homotopies above together with ambient isotopies. We call such a homotopy a *generic regular homotopy*.

Since two plane curves are regular homotopic if and only if these curves have the same rotation number (see [14]), any plane curve with the rotation number $\pm i$ is regular homotopic to the plane curve e_i (up to orientations) shown in Figure 2.3.

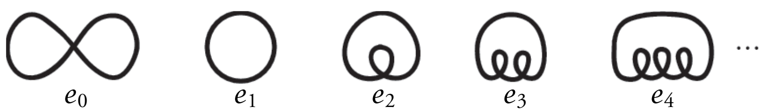

FIGURE 2.3. The base curve e_i of the rotation number $\pm i$.

Theorem 2.1 ([3]). *Denote by n_i the number of double points of e_i. For a plane curve $C \subset \mathbb{R}^2$ with the rotation number i, we take a generic regular homotopy $H : S^1 \times [0,1] \to \mathbb{R}^2$ from e_i to C. Using H, we assign three integers $J^+(C)$, $J^-(C)$ and $St(C)$ to C as follows:*

$$\begin{aligned} J^+(C) &= \min\{0, -2(|i|-1)\} + 2(d_+ - d_-), \\ J^-(C) &= \min\{0, -2(|i|-1)\} - n_i - 2(i_+ - i_-), \\ St(C) &= \max\{0, |i|-1\} + t_+ - t_-, \end{aligned}$$

where $d_\pm$, $i_\pm$ and $t_\pm$ are the numbers of positive/negative dR2*-moves,* iR2*-moves, and* R3*-moves in H, respectively. Then, the integers $J^\pm(C)$ and $St(C)$ do not depend on the choice of a generic regular homotopy H.*

By Theorem 2.1 the integers $J^\pm(C)$ and $St(C)$ are invariants of (an isotopy class of) a plane curve C. These invariants are called the *Arnold invariants* for plane curves.

We can also define similar invariants for spherical curves. For a spherical curve $C \subset S^2$, we take a point $x \in S^2 \setminus C$, and denote by $C_x \subset \mathbb{R}$ the image of C under the stereographic projection from x. We assign integers $J_S^\pm(C)$ and $St_S(C)$ as follows:

$$J_S^+(C) = J^+(C_x) + \frac{\mathrm{rot}(C_x)^2}{2},$$
$$J_S^-(C) = J^-(C_x) + \frac{\mathrm{rot}(C_x)^2}{2},$$
$$St_S(C) = St(C_x) - \frac{\mathrm{rot}(C_x)^2}{4}.$$

It is known that the integers $J_S^\pm(C)$ and $St_S(C)$ do not depend on the choice of a point x (cf [5]). For this reason, these are invariants for (isotopy classes of) spherical curves, which are also called the *Arnold invariants* for spherical curves.

3. LEGENDRIAN KNOTS IN THE UNIT TANGENT BUNDLE OF S^2

Denote by $UTS^2 \subset TS^2$ the unit tangent bundle of S^2. In this section, we give an algorithm to obtain a diagram of the Legendrian knot in UTS^2 associated with an oriented curve.

We begin by reviewing Legendrian knots. For an oriented spherical curve C, we take a generic immersion $f : S^1 \to S^2$ so that the image of f is C and so that the orientation of C induced by that of S^1 coincides with the given orientation. We denote the derivative of f by $df : TS^1 \to TS^2$. Since f is an immersion, $df(p)$ is everywhere non-zero, and we can compose the projection $\pi : TS^2 \setminus S^2 \to UTS^2$ to df, where we identify S^2 with the 0-section of TS^2. Since f is generic, the composition $\pi \circ df : S^1 \to UTS^2$ is an embedding, where we identify the set of unit positive vectors with S^1 so that we can regard S^1 as a subset of TS^1. The image of the composition $\pi \circ df$ is a knot in UTS^2. This knot, which is denoted by K_C, is called the *Legendrian knot associated with C*.

Remark 3.1 (The canonical framing of K_C and its relation with J^+). Since K_C is a Legendrian knot with respect to the canonical contact structure of UTS^2, this knot has the canonical framing that is induced by a vector field everywhere transverse to the contact plane. This framing can be obtained in the following way: we can take a vector field V on $C \subset S^2$ so that, for any point $p \in C$, V_p and the unit positive tangent vector of C at p span the tangent space of S^2. This vector field can be lifted to that of K_C via $d\pi$. A parallel shift of K_C along this lift is a framing of K_C. It is easy to see that this framing does not depend on the choice of V and its lift, and coincides with the canonical framing of K_C.

Arnold [3,4] proved that, for a plane curve C, the invariant $J^+(C)$ is equal to $1 - \beta(K_C)$, where $\beta(K_C)$ is the *Bennequin-Tabachnikov number* of the Legendrian knot K_C introduced in [12]. This invariant is defined as follows: for a Legendrian knot L in $UT\mathbb{R}^2$, we take points $p_1, \ldots, p_s \in \mathbb{R}^2$ sufficiently far from the origin and a 2-chain Δ bounding the union $L \cup (\bigsqcup_i p^{-1}(p_i))$, where $p : UT\mathbb{R}^2 \to \mathbb{R}^2$ is the projection. Then, $\beta(L)$ is defined to be the intersection number between Δ and a parallel shift of L along the canonical framing.

It is easy to prove the following proposition by the local pictures of R3-moves and iR2-moves.

Proposition 3.2. *If two spherical curves $C_0, C_1 \subset S^2$ are equivalent under* $\{\text{iR2}, \text{R3}\}$, *then the corresponding Legendrian knots K_{C_0} and K_{C_1} are Legendrian isotopic; in particular, these are isotopic (as framed knots) in UTS^2.*

Remark 3.3. It is also easy to prove that a dR2-move for a curve in S^2 corresponds with a crossing change to the corresponding Legendrian knot.

The projection $p : UTS^2 \to S^2$ is an S^1-bundle over S^2 whose Euler number is 2 (the orientation of UTS^2 is derived from that of S^2). Thus, the manifold UTS^2 has a surgery diagram which consists of an unknot with framing 2. For a curve $C \subset \mathbb{R}^2 \subset S^2$, we take a sufficiently large disk $D \subset \mathbb{R}^2$ that contains C. The restriction $p|_{p^{-1}(D)} : p^{-1}(D) \to D$ is the trivial S^1-bundle, and the submanifold $p^{-1}(D)$ coincides with the complement of the framed unknot (which is a solid torus) in the surgery diagram of UTS^2. We fix a trivialization $p^{-1}(D) \cong D \times S^1$, and call each disk $D \times \{*\}$ a *sectional disk*. We take an identification of UTS^2 with the manifold described by the diagram $\bigcirc^{+2}$, so that the set of horizontal vectors in $D \subset \mathbb{R}^2$ oriented from right to left coincides with the sectional disk that contains the infinity $\infty \in S^3$. The fiber of UTS^2 at a point $p_0 \in D$ is the set of unit tangent vectors at p_0, and this fiber is oriented counterclockwise since the orientation of UTS^2 is derived from that of S^2. We identify this fiber with the unit circle $S^1 \subset \mathbb{C}$ in the obvious way. We can assume that the set $\{\exp(\sqrt{-1}\theta) \in S^1 \mid \theta \in (-\pi + \varepsilon, \pi - \varepsilon)\} \subset S^1 \cong p^{-1}(p_0)$ is projected to a point q_0 in the diagram $\bigcirc^{+2}$ for a sufficiently small $\varepsilon > 0$. It is easy to see that, for $\theta_0, \theta_1 \in (-\pi + \varepsilon, \pi - \varepsilon)$, a point $\exp(\sqrt{-1}\theta_0)$ is behind a point $\exp(\sqrt{-1}\theta_1)$ in the diagram $\bigcirc^{+2}$ if $\theta_0 < \theta_1$. Eventually, we can obtain a knot diagram of K_C in the surgery diagram $\bigcirc^{+2}$ in the following way:

- A knot projection that is derived from K_C coincides with a diagram of $C \subset D \subset \mathbb{R}^2$ except in neighborhoods of points at which C has a horizontal direction vector oriented from right to left, and this knot projection is drawn inside the $(+2)$-framed unknot.
- At each double point q of the knot projection, the curve with direction vector $\exp(\sqrt{-1}\theta_0)$ goes behind the other curve with direction vector

$\exp(\sqrt{-1}\theta_1)$ if $\theta_0, \theta_1 \in (-\pi + \varepsilon, \pi - \varepsilon)$ and $\theta_0 < \theta_1$, where we identify the fiber of $UTS^2|_D$ with the unit circle $S^1 \subset \mathbb{C}$ in the same way as above.

- In the preimage (under p) of a neighborhood of each point at which C has a horizontal direction vector oriented from right to left, K_C travels along the fiber of the projection $p : UTS^2 \to S^2$ once, following the rule shown in Figure 3.1.

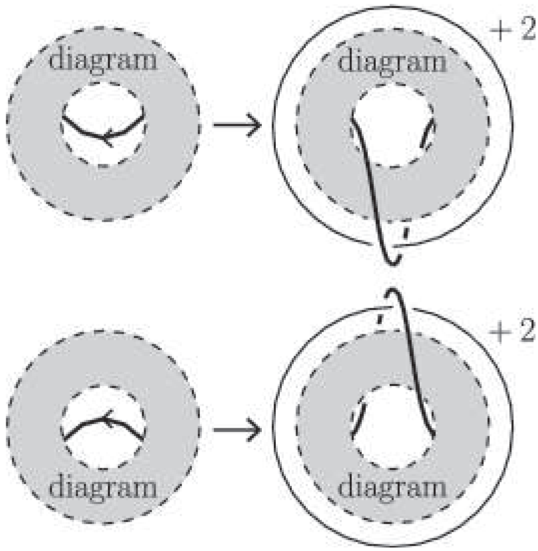

FIGURE 3.1. Left: diagrams of C in $\mathbb{R}^2$. Right: knot diagrams of K_C in the surgery diagram $\bigcirc^{+2}$.

Note that the framing of K_C is along sectional disks. An example of a pair of a spherical curve and the corresponding framed knot is shown in Figure 3.2.

By regarding $\mathbb{R}^2$ as a subset of $S^2 = \mathbb{R}^2 \cup \{\infty\}$, we can thus obtain a knot $K_C \subset UTS^2$ and a diagram of K_C drawn in the surgery diagram $\bigcirc^{+2}$. Under the identification given in the previous paragraph, when a curve C is deformed by a sequence of R3-moves, iR2-moves, and isotopies in $\mathbb{R}^2$ (namely, isotopies in S^2 fixing the point $\infty \in S^2$), the associated knot K_C is deformed by isotopies in UTS^2 avoiding the core of solid torus attached by the surgery. This observation yields the following proposition.

Proposition 3.4. *If there are two plane curves $C_0, C_1 \subset \mathbb{R}^2$ equivalent under* $\{$iR2, R3$\}$, *then there exists an ambient isotopy in UTS^2 that deforms K_{C_0} to K_{C_1} and keeps the torus attached by the surgery fixed.*

Since the Legendrian knot K_C is contained in the complement of a regular neighborhood of $(+2)$-framed unknot, which is regarded as a subset of S^3, we can

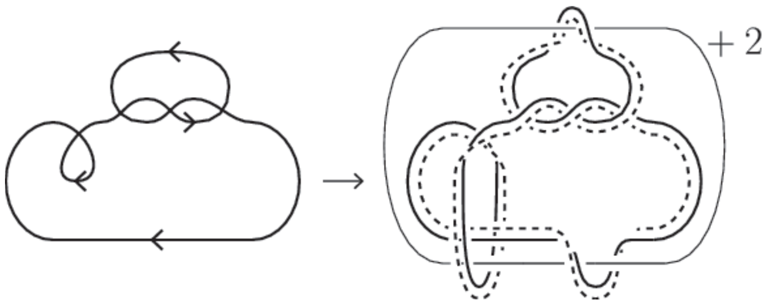

FIGURE 3.2. Left: a spherical curve. Right: the associated knot in UTS^2. The dotted curve describes the framing.

regard K_C as a framed knot in S^3. From Proposition 3.4, we immediately obtain the following corollary.

Corollary 3.5. *Under the same assumption as in Proposition 3.4, there exists an ambient isotopy in S^3 that deforms K_{C_0} to K_{C_1} and keeps the $(+2)$-framed unknot fixed. In particular, K_{C_0} and K_{C_1} are isotopic as framed knots in S^3.*

4. The Universal Cover of UTS^2 and Knot Diagrams

Since the Euler number of the S^1-bundle $p : UTS^2 \to S^2$ is 2, and in particular UTS^2 is diffeomorphic to $\mathbb{RP}^3$, the universal cover of UTS^2 is a double cover from S^3, which we denote by $q : S^3 \to UTS^2$. In this section, we explain how to obtain a link diagram of the preimage of a knot K in UTS^2 under q from a knot diagram of K in the surgery diagram $\bigcirc^{+2}$.

For a knot K in UTS^2, we denote the preimage of K under q by $\tilde{K} \subset S^3$. Note that $\tilde{K}$ is a 2-component link if K is null-homologous in UTS^2, and a knot otherwise. Any knot K has a knot diagram in the surgery diagram $\bigcirc^{+2}$ as shown in the left side of Figure 4.1 (in other words, any knot in UTS^2 can be obtained by taking band sums between a knot contained in a small ball in UTS^2 and some meridians of the framed unknot).

The composition $p \circ q : S^3 \to S^2$ is an S^1-bundle over S^2 whose Euler number is 1. Since the covering map $q : S^3 \to UTS^2$ preserves fibers, the covering transformation $T : S^3 \to S^3$ acts on each fiber of $p \circ q$ by multiplication of $-1 \in S^1$. Thus, we can obtain a diagram of $\tilde{K}$ as shown in the right side of Figure 4.1, where the shaded box outside the framed unknot contains a diagram of the image of a tangle contained in the shaded box inside the unknot under T.

Both the diagram $\bigcirc^{+1}$ and the empty diagram describe S^3, and these diagrams are related by blowing down. Thus, in order to obtain a usual link diagram of $\tilde{K}$ (i.e., a diagram of $\tilde{K} \subset S^3$ derived from the projection $S^3 \setminus \{\infty\} \to \mathbb{R}^2$), we have to blow down $\tilde{K}$ along the $(+1)$-framed unknot. In Figure 4.2, we describe

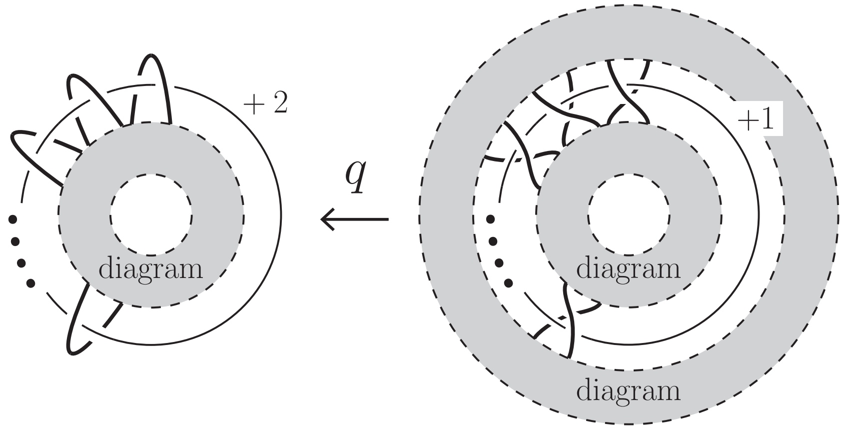

FIGURE 4.1. Left: a diagram of K in UTS^2. Right: a diagram of the preimage under q.

a diagram of the preimage of the knot in the right side of Figure 3.2 under q, obtained by the above procedure.

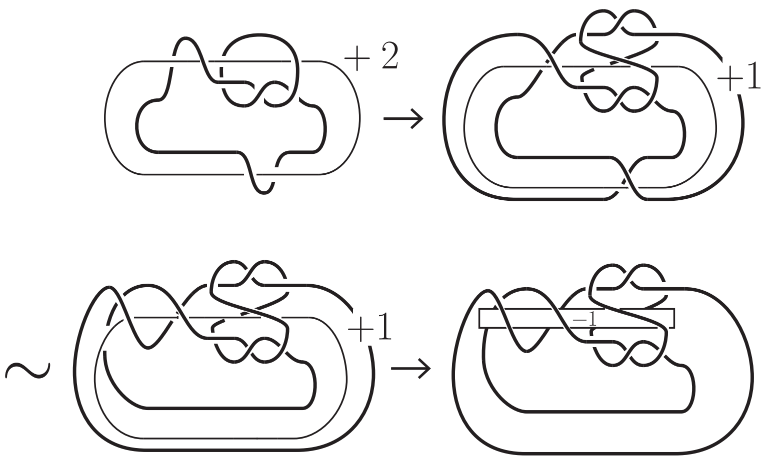

FIGURE 4.2. An example of the preimage of a knot in UTS^2 under q.

5. INFINITELY MANY CURVES WITH THE SAME VALUES OF THE INVARIANTS

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We first note it is sufficient to prove the statement of Theorem 1.1 for non-negative integer r. Indeed, if a family of oriented plane curves satisfies the desired conditions, the family consisting of the same curves with the opposite orientations also satisfies the conditions in Theorem 1.1. For this reason, we only give a proof for the case where r is non-negative.

For non-negative integers a, b, and c, we define a curve $C(a, b, c)$ by Figure 5.2, where tangles (a), (b), and (c) are defined by the local figures in Figure 5.1, and $x = a$, b, or c is the number of tangles labeled by (x).

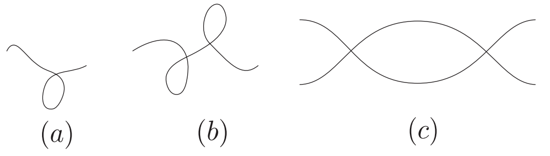

FIGURE 5.1. Tangles (a), (b) and (c)

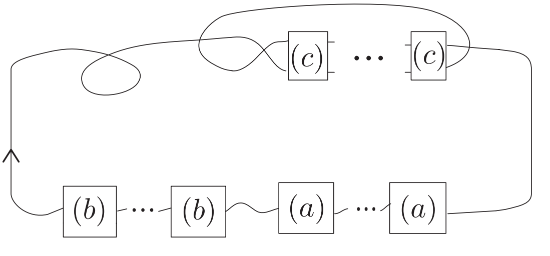

FIGURE 5.2. $C(a, b, c)$

The rotation number of $C(a, b, c)$ is $a - 1$. Moreover, it is easy to see that $J^+(C(a, b+1, c)) - J^+(C(a, b, c))$ is equal to -2, while $J^+(C(a, b, c+1)) - J^+(C(a, b, c))$ is equal to 2 for any $a, b, c \geq 0$. Thus, we can find non-negative numbers b_0 and c_0 such that the value $J^+(C(r+1, b_0, c_0))$ is equal to j^+. Furthermore, by the observation above, all the curves in the family

$$\{C(r+1, b_0+k, c_0+k)\}_{k \geq 0}$$

have the same rotation number and the same value of the invariant J^+.

We can obtain the diagram of $K_{C(r+1, b_0+k, c_0+k)}$, by the algorithm given in Section 3, as shown in Figure 5.3.

The diagram shows that $K_{C(r+1, b_0+k, c_0+k)}$ is the $(2, 2(c_0+k)+3)$-torus knot, where we regard this knot as that in $\mathbb{R}^3$ (see Proposition 3.4 and the paragraph preceding it); in particular, $K_{C(r+1, b_0+k_1, c_0+k_1)}$ and $K_{C(r+1, b_0+k_2, c_0+k_2)}$ are isotopic if and only if $k_1 = k_2$. By Proposition 3.4, we have that any two curves in the family $\{C(r+1, b_0+k, c_0+k)\}_{k \geq 0}$ are not equivalent under $\{\mathrm{iR2}, \mathrm{R3}\}$. This completes the proof of Theorem 1.1. $\square$

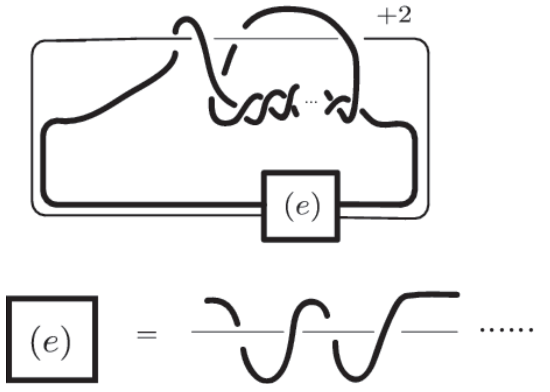

FIGURE 5.3. The diagram of $K_{C(r+1,b_0+k,c_0+k)}$.

We next give a proof of Theorem 1.2.

Proof of Theorem 1.2. By regarding S^2 as the one-point compactification of $\mathbb{R}^2$, we think of the curve $C(a,b,c)$ as a spherical curve. As in the proof of Theorem 1.1, it is easy to verify that $J_S^+(C(a,b+1,c)) - J_S^+(C(a,b,c))$ is equal to -2, while $J_S^+(C(a,b,c+1)) - J_S^+(C(a,b,c))$ is equal to 2 for any $a,b,c \geq 0$.

We first prove the statement for the case $r = 0$. By the observation above, we can find non-negative integers b_1 and c_1 such that the value $J_S^+(C(1,b_1,c_1))$ is equal to j^+. Furthermore, the value of the invariant J_S^+ is the same for all the curves in the family $\{C(1,b_1+k,c_1+k)\}_{k\geq 0}$. The diagram of the preimage $q^{-1}(K_{C(1,b_1+k,c_1+k)})$ under the universal cover $q : S^3 \to UTS^2$ obtained by the algorithm in Section 4 is described in Figure 5.4.

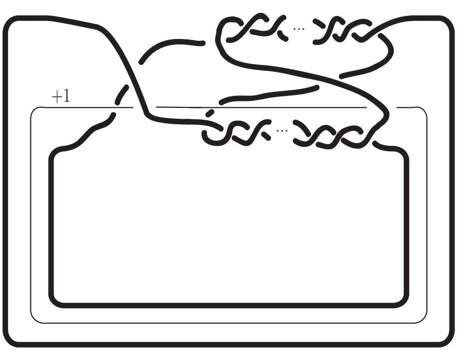

FIGURE 5.4. The diagram of $K_{C(1,b_1+k,c_1+k)}$.

It is easy to see that the linking number between the two components of $q^{-1}(K_{C(1,b_1+k,c_1+k)})$ is equal to $-2(k+c_1)$. Thus, $q^{-1}(K_{C(1,b_1+k_1,c_1+k_1)})$ and

$q^{-1}(K_{C(1,b_1+k_2,c_1+k_2)})$ are isotopic if and only if $k_1 = k_2$. Since any isotopy between two links in UTS^2 can be lifted to that of the preimage of them under q, $K_{C(1,b_1+k_1,c_1+k_1)}$ and $K_{C(1,b_1+k_2,c_1+k_2)}$ are isotopic if and only if $k_1 = k_2$. By Proposition 3.2, any two spherical curves in the family $\{C(1, b_1 + k, c_1 + k)\}_{k\geq 0}$ are not equivalent under $\{\mathrm{iR2}, \mathrm{R3}\}$.

We next prove the statement for the case $r = 1$. In the same way as in the previous paragraph, we can find non-negative integers b_2 and c_2 such that the value $J_S^+(C(2, b_2 + k, c_2 + k))$ is equal to j^+ for any $k \geq 0$. The diagram of the preimage $q^{-1}(K_{C(2,b_2+k,c_2+k)})$ obtained by the algorithm in Section 4 is described in Figure 5.5. We can deduce from this diagram that $q^{-1}(K_{C(2,b_2+k,c_2+k)})$ is the

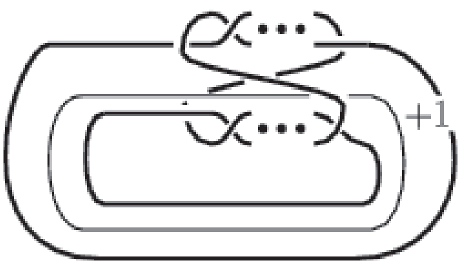

FIGURE 5.5. A diagram of the preimage $q^{-1}(K_{C(2,b_2+k,c_2+k)})$.

connected sum of two $(2, 2(c_2 + k) + 3)$-torus knots. The Jones polynomial of this knot is equal to the square of the polynomial of the $(2, 2(c_2 + k) + 3)$-torus knot, which is equal to $t^{-(c_2+k)-1} - t^{-(c_2+k)-4} - t^{-3(c_2+k)-5} + t^{-3(c_2+k)-6}$.

In particular, $q^{-1}(K_{C(2,b_2+k_1,c_2+k_1)})$ and $q^{-1}(K_{C(2,b_2+k_2,c_2+k_2)})$ are isotopic if and only if $k_1 = k_2$. Thus, any two curves in the family $\{C_{(2,b_2+k,c_2+k)}\}_{k\geq 0}$ are not equivalent under $\{\mathrm{iR2}, \mathrm{R3}\}$. This completes the proof of Theorem 1.2. ◻

Remark 5.1. For non-negative integers a, b, c, the Arnold invariants of the plane curve $C(a, b, c)$ are calculated as follows:

$$J^+(C(a,b,c)) = -2b + 2c,$$
$$J^-(C(a,b,c)) = -4b - a - 4,$$
$$St(C(a,b,c)) = b + 1.$$

In particular, any two curves in the family $\{C(r+1, b_0+k, c_0+k)\}_{k\geq 0}$ (constructed in the proof of Theorem 1.1) have different values of the invariants J^- and St.

As explained in Section 3, we can regard K_C as an oriented knot in S^3 for a plane curve C. We denote the framing coefficient of the canonical framing of K_C by $\mathrm{fr}(C)$, and the linking number of K_C with the $(+2)$-framed unknot by $\mathrm{lk}(C)$. These numbers are clearly invariants of equivalent classes of plane curves under $\{\mathrm{iR2}, \mathrm{R3}\}$.

Corollary 5.2. *As an invariant of equivalent classes of plane curves under* $\{\mathrm{iR2}, \mathrm{R3}\}$*, the pair* $(\mathrm{fr}(C), \mathrm{lk}(C), V_{K_C})$ *is stronger than* (rot, J^+)*, where* V_{K_C} *is the Jones polynomial of* K_C*.*

Proof. It is sufficient to prove that $\mathrm{fr}(C)$ and $\mathrm{lk}(C)$ are equal to $\beta(K_C)$ and $\mathrm{rot}(C)$, respectively, where $\beta(K_C)$ is the Bennequin-Tabachnikov number of K_C (see Remark 3.1). We take a Seifert surface $\Sigma \subset S^3$ of K_C intersecting the $(+2)$-framed unknot K_0 transversely. The framing coefficient $\mathrm{fr}(K_C)$ is equal to the intersection number $\tilde{K}_C \cdot \Sigma$, where $\tilde{K}_C$ is a parallel shift of K_C along the canonical framing. On the other hand, by the definition, we have that $\beta(K_C)$ is equal to $\tilde{K}_C \cdot (\Sigma \setminus (\bigsqcup_q D_q))$, where D_q is a small disk neighborhood of $q \in \Sigma \cap K_0$ in Σ. Thus, $\mathrm{fr}(C)$ is equal to $\beta(K_C)$.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the vertical axis. We denote by L_M (respectively, L_m) the set of local maxima (respectively, minima) of the restriction $\pi|_C$ at which C goes from right to left with respect to the given orientation. It is not hard to see that the rotation number of C is equal to $\sharp L_M - \sharp L_m$. According to the algorithm, to obtain the knot K_C from C given in Section 3, $\mathrm{lk}(C)$ is also equal to $\sharp L_M - \sharp L_m$ (see Figure 3.1). □

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