

習題 1.2.2

$$\square (1) \frac{dy}{dx} = \frac{2xy}{x^2+y^2} = \frac{2(\frac{y}{x})}{1+(\frac{y}{x})^2} \quad v = \frac{y}{x} \text{ 則 } y = xv \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{代入上式: } v + x \frac{dv}{dx} = \frac{2v}{1+v^2} \therefore x \frac{dv}{dx} = \frac{2v}{1+v^2} - v = \frac{2v - v - v^3}{1+v^2}$$

$$= \frac{v - v^3}{1+v^2} \therefore \frac{1+v^2}{v(1-v^2)} dv = \frac{dx}{x} \therefore \int \left(\frac{1}{v} + \frac{2v}{1-v^2} \right) dv = \log x + c$$

$$\therefore \log v - \log(1-v^2) = \log x + c \therefore \log \frac{v}{1-v^2} = \log e^c \cdot x \therefore \frac{v}{1-v^2} = e^c \cdot x$$

$$\therefore \frac{v}{1-v^2} = cx \therefore \frac{xy}{x^2-y^2} = cx \therefore x^2-y^2 = \frac{1}{c} y \therefore x^2-y^2 = cy$$

$$(2) xy \frac{dy}{dx} = y^2 - x^2 \therefore \frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{(\frac{y}{x})^2 - 1}{\frac{y}{x}} \quad v = \frac{y}{x} \text{ 則 } y = xv \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{代入上式: } v + x \frac{dv}{dx} = \frac{v^2 - 1}{v} = v - \frac{1}{v} \therefore x \frac{dv}{dx} = -\frac{1}{v} \therefore v dv = -\frac{dx}{x}$$

$$\therefore \frac{1}{2} v^2 = -\log x + c \therefore v^2 = -2 \log x + 2c \therefore \frac{y^2}{x^2} = -2 \log x + 2c \therefore \frac{y^2}{x^2} = -2 \log x + c$$

$$\therefore y^2 = x^2 (-2 \log x + c)$$

$$(3) \frac{dy}{dx} = \frac{x+y}{y-x} = \frac{1+\frac{y}{x}}{\frac{y}{x}-1} \quad v = \frac{y}{x} \text{ 則 } y = xv \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{代入上式: } v + x \frac{dv}{dx} = \frac{v+1}{v-1} \therefore x \frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{v+1-v^2-v}{v-1} = -\frac{v^2-2v-1}{v-1}$$

$$\therefore \frac{v-1}{v^2-2v-1} dv = -\frac{dx}{x} \therefore \frac{1}{2} \log(v^2-2v+1) = -\log x + c \therefore \log(v^2-2v-1) = \log \frac{e^{2c}}{x^2}$$

$$\therefore v^2-2v-1 = \frac{e^{2c}}{x^2} \therefore y^2-2xy-x^2 = e^{2c} \therefore x^2+2xy-y^2 = c$$

$$(4) \frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x} \quad v = \frac{y}{x} \text{ 則 } y = xv \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = v + \tan v \therefore \frac{\tan v}{\sin v} dv = \frac{dx}{x} \therefore \log \sin v = \log x + c = \log e^c \cdot x$$

$$\therefore \sin v = e^c \cdot x \therefore \sin \frac{y}{x} = cx$$

$$(5) \frac{dy}{dx} = \left(\frac{y}{x} \right)^2 + \frac{y}{x} + 1 \quad v = \frac{y}{x} \text{ 則 } y = xv \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$V + x \frac{dV}{dx} = V^2 + V + 1 \quad \therefore \frac{dV}{V^2+1} = \frac{dx}{x} \quad \therefore \tan^{-1} V = \log x + c \quad \therefore V = \tan(\log x + c)$$

$$\therefore y = x \tan(\log x + c)$$

$$(6) \frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \quad V = \frac{y}{x} \text{ एतक } \frac{dy}{dx} = V + x \frac{dV}{dx} \quad \text{अब इसी तरह } x:$$

$$x + x \frac{dV}{dx} = x + \sqrt{1 + V^2} \quad \therefore \frac{dV}{\sqrt{1+V^2}} = \frac{dx}{x} \quad \therefore \log(V + \sqrt{V^2+1}) = \log x + c$$

$$\therefore V + \sqrt{V^2+1} = e^c \cdot x \quad \therefore \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = Cx \quad \therefore y + \sqrt{x^2 + y^2} = Cx^2$$

$$\boxed{2} (1) \text{ दो रेखाएँ } 2x - y - 1 = 0, x - 2y + 3 = 0 \text{ का प्रतिच्छेद बिंदु } (x, y) = \left(\frac{5}{3}, \frac{7}{3}\right). \text{ जो } p = x - \frac{5}{3}, q = y - \frac{7}{3}.$$

$$\text{एतक } x \text{ एत } \frac{dy}{dx} = \frac{dq}{dp} \text{ ठीक इसी तरह } \frac{dq}{dp} = \frac{2p - q}{p - 2q} = \frac{2 - \frac{q}{p}}{1 - 2\left(\frac{q}{p}\right)} \quad V = \frac{q}{p} \text{ एतक}$$

$$\frac{dq}{dp} = V + p \frac{dV}{dp} \quad \text{अब इसी तरह } V + p \frac{dV}{dp} = \frac{2-V}{1-2V} \quad \therefore \frac{2V-1}{2V^2-2V+2} dV = -\frac{dp}{p}$$

$$\therefore -\frac{1}{2} \log(2V^2 - 2V + 2) = -\log p + c \quad \therefore \log(2V^2 - 2V + 2) = \log \frac{e^{2c}}{p^2}$$

$$\therefore V^2 - V + 1 = \frac{C}{p^2} \quad \therefore p^2 - pq + q^2 = C \quad \therefore \left(x - \frac{5}{3}\right)^2 - \left(x - \frac{5}{3}\right)\left(y - \frac{7}{3}\right) + \left(y - \frac{7}{3}\right)^2 = C$$

$$(2) \text{ दो रेखाएँ } 6x - 2y - 3 = 0, 2x + 2y - 1 = 0 \text{ का प्रतिच्छेद बिंदु } (x, y) = \left(\frac{1}{2}, 0\right). \text{ जो } p = x - \frac{1}{2}, q = y$$

$$p = x - \frac{1}{2}, q = y \text{ एतक } \frac{dy}{dx} = \frac{dq}{dp} \text{ ठीक इसी तरह } \frac{dq}{dp} = \frac{6p - 2q}{2p + 2q} = \frac{3p - q}{p + q} = \frac{3 - \frac{q}{p}}{1 + \frac{q}{p}} \quad V = \frac{q}{p} \text{ एतक } \frac{dq}{dp} = V + p \frac{dV}{dp} \quad \text{अब इसी तरह}$$

$$V + p \frac{dV}{dp} = \frac{3-V}{1+V} \quad \therefore \frac{V+1}{V^2+2V-3} dV = -\frac{dp}{p} \quad \therefore \frac{1}{2} \log(V^2+2V-3) = -\log p + c$$

$$\therefore \log(V^2+2V-3) = \log \frac{e^{2c}}{p^2} \quad \therefore V^2+2V-3 = \frac{C}{p^2} \quad \therefore q^2+2pq-3p^2 = C$$

$$\therefore y^2 + 2y\left(x - \frac{1}{2}\right) - 3\left(x - \frac{1}{2}\right)^2 = C$$

$$\boxed{3} \quad x^2 + y^2 = cx \quad \therefore 2x + 2yy' = c \quad \therefore x^2 + y^2 = x(2x + 2yy')$$

$$\therefore x^2 + y^2 = 2x^2 + 2xyy' \quad \therefore 2xyy' = y^2 - x^2 \quad \text{Divide both sides by } x^2$$

$$2xy \left(-\frac{1}{y^2}\right) = y^2 - x^2 \quad \therefore \frac{y'}{2xy} = \frac{1}{x^2 - y^2} \quad \therefore y' = \frac{2xy}{x^2 - y^2} = \frac{2\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^2}$$

$$v = \frac{y}{x} \quad \text{Let } v = \frac{y}{x} \quad \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{Substitute } v = \frac{y}{x}, \quad v + x \frac{dv}{dx} = \frac{2v}{1 - v^2}$$

$$\therefore x \frac{dv}{dx} = \frac{v^2 + v}{1 - v^2} \quad \therefore \frac{v^2 - 1}{v(v^2 + 1)} dv = -\frac{dx}{x} \quad \therefore \int \left(\frac{2v}{v^2 + 1} - \frac{1}{v} \right) dv = -\log x + c$$

$$\therefore \log(v^2 + 1) - \log v = -\log x + c \quad \therefore x \left(\frac{v^2 + 1}{v} \right) = e^c \quad \therefore x^2 + y^2 = e^c \cdot y$$

$$\therefore x^2 + y^2 = cy \quad \text{Put } x=1, y=1 \text{ in } 1^2 + 1^2 = c \cdot 1 \quad \therefore c=2 \quad \therefore \underline{x^2 + y^2 = 2y}$$