演習1.2.1

$\boxed{1}$ (1) $x\dfrac{dy}{dx}=-(y+1)\quad \therefore \dfrac{dy}{y+1}=-\dfrac{dx}{x}\quad \therefore \log(y+1)=-\log x+c=\log\dfrac{e^c}{x}$

$\therefore y+1=\dfrac{e^c}{x}\quad \therefore y=\dfrac{c}{x}-1$

(2) $(y+1)\dfrac{dy}{dx}=1-x\quad \therefore (y+1)dy=(1-x)dx\quad \therefore \dfrac{1}{2}(y+1)^2=x-\dfrac{x^2}{2}+c$

$\therefore (y+1)^2=2x-x^2+2c\quad \therefore (y+1)^2=2x-x^2+c$

(3) $\dfrac{dy}{dx}=(\tan y)(\tan x)\quad \therefore \dfrac{dy}{\tan y}=\tan x\,dx\quad \therefore \dfrac{\cos y}{\sin y}dy=\dfrac{\sin x}{\cos x}dx$

$\therefore \displaystyle\int\frac{\cos y}{\sin y}dy=\int\frac{\sin x}{\cos x}dx+c\quad \therefore \log\sin y=-\log\cos x+c=\log\frac{e^c}{\cos x}$

$\therefore \sin y=\dfrac{e^c}{\cos x}\quad \therefore \sin y=\dfrac{c}{\cos x}$

(4) $xy(1+x^2)\dfrac{dy}{dx}=1+y^2\quad \therefore \dfrac{y}{1+y^2}dy=\dfrac{dx}{x(1+x^2)}$

$\therefore \dfrac{1}{2}\log(1+y^2)=\displaystyle\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx+c=\log x-\frac{1}{2}\log(1+x^2)+c=\log\frac{e^c\cdot x}{\sqrt{1+x^2}}$

$\therefore \log\sqrt{1+y^2}=\log\dfrac{e^c\cdot x}{\sqrt{1+x^2}}\quad \therefore \sqrt{1+y^2}=\dfrac{e^c\cdot x}{\sqrt{1+x^2}}\quad \therefore 1+y^2=\dfrac{(e^c)^2x^2}{1+x^2}$

$\therefore (1+x^2)(1+y^2)=e^{2c}\cdot x^2\quad \therefore (1+x^2)(1+y^2)=cx^2$

(5) $(1+x)y+2(1-y)x\dfrac{dy}{dx}=0\quad \therefore 2(y-1)x\dfrac{dy}{dx}=(x+1)y$

$\therefore \dfrac{y-1}{y}dy=\dfrac{x+1}{2x}dx\quad \therefore \left(1-\dfrac{1}{y}\right)dy=\dfrac{1}{2}\left(1+\dfrac{1}{x}\right)dx$

$\therefore \displaystyle\int\left(1-\frac{1}{y}\right)dy=\frac{1}{2}\int\left(1+\frac{1}{x}\right)dx+c\quad \therefore y-\log y=\frac{1}{2}(x+\log x)+c$

$\therefore 2y-2\log y=x+\log x+2c\quad \therefore \log xy^2=2y-x-2c\quad \therefore xy^2=e^{2y-x-2c}$

$\therefore xy^2=e^{-2c}\cdot e^{2y-x}\quad \therefore xy^2=ce^{2y-x}$

(6) $y\dfrac{dy}{dx}=xe^{x^2}\cdot e^{y^2}\quad \therefore ye^{-y^2}dy=xe^{x^2}dx\quad \therefore \displaystyle\int ye^{-y^2}dy=\int xe^{x^2}dx+c$

$\therefore -\dfrac{e^{-y^2}}{2}=\dfrac{e^{x^2}}{2}+c\quad \therefore -e^{-y^2}=e^{x^2}+2c\quad \therefore e^{x^2}+e^{-y^2}=c$

(7) $(1-x^2)\frac{dy}{dx}=y^2-1$ ∴ $\frac{dy}{y^2-1}=-\frac{dx}{x^2-1}$

∴ $\frac{1}{2}\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy=-\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx+c$

∴ $\frac{1}{2}\log\frac{y-1}{y+1}=-\frac{1}{2}\log\frac{x-1}{x+1}+c$ ∴ $\log\frac{y-1}{y+1}=\log e^{2c}\cdot\frac{x+1}{x-1}$

∴ $\frac{y-1}{y+1}=e^{c}\cdot\frac{x+1}{x-1}$ ∴ $\frac{y-1}{y+1}=c\cdot\frac{x+1}{x-1}$ ∴ $(y-1)(x-1)=c(x+1)(y+1)$

∴ $xy-y-x+1=c(xy+x+y+1)=cxy+cx+cy+c$

∴ $xy-y-cxy-cy=cx+c+x-1$ ∴ $\{(1-c)x-(1+c)\}y=(c+1)x+(c-1)$

∴ $y=\frac{(1+c)x-(1-c)}{(1-c)x-(1+c)}=\frac{x-\frac{1-c}{1+c}}{\frac{1-c}{1+c}x-1}$ ∴ $y=\frac{x-c}{cx-1}$

(8) $\frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ ∴ $\frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$ ∴ $\int\frac{dy}{\sqrt{1-y^2}}=-\int\frac{dx}{\sqrt{1-x^2}}+c$

∴ $\sin^{-1}y+\sin^{-1}x=c$ ここで、$u=\sin^{-1}x$, $v=\sin^{-1}y$ とおくと、$x=\sin u$, $y=\sin v$.

∴ $\sin(u+v)=\sin u\cos v+\cos u\sin v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

∴ $x\sqrt{1-y^2}+y\sqrt{1-x^2}=\sin c$ ∴ $x\sqrt{1-y^2}+y\sqrt{1-x^2}=c$

$\boxed{2}$ (1) $u=y-x$ とおくと $\frac{du}{dx}=y'-1$ ∴ $1+\frac{du}{dx}=u^2$ ∴ $\frac{du}{u^2-1}=dx$

∴ $\frac{1}{2}\log\frac{u-1}{u+1}=x+c$ ∴ $\log\frac{u-1}{u+1}=2x+2c=\log e^{2c}\cdot e^{2x}$ ∴ $\frac{u-1}{u+1}=e^{2c}\cdot e^{2x}$

∴ $\frac{u-1}{u+1}=c_1e^{2x}$ ∴ $\frac{y-x-1}{y-x+1}=c_1e^{2x}$ ∴ $y-x-1=c_1e^{2x}y-c_1xe^{2x}+c_1e^{2x}$

∴ $(1-c_1e^{2x})y=x+1-c_1xe^{2x}+c_1e^{2x}$

∴ $(1-c_1e^{2x})y=x(1-c_1e^{2x})+1+c_1e^{2x}$ ∴ $y=x+\frac{1+c_1e^{2x}}{1-c_1e^{2x}}$

∴ $y=x+\frac{1+ce^{2x}}{1-ce^{2x}}$

(2) $u=x+e^y$ とおくと $\frac{du}{dx}=1+e^y\cdot y'$ ∴ $\frac{du}{dx}-1=u-1$ ∴ $\frac{du}{dx}=u$ ∴ $\frac{du}{u}=dx$

∴ $\log u=x+c=\log e^c\, e^x$ ∴ $u=e^c\cdot e^x$ ∴ $u=c\cdot e^x$ ∴ $x+e^y=ce^x$

∴ $e^y=ce^x-x$ ∴ $y=\log(ce^x-x)$

(3) $u=xy$ とおくと $\frac{du}{dx}=y+xy'$ ∴ $xy'=\frac{du}{dx}-y$

∴ $y'=\frac{1}{x}\frac{du}{dx}-\frac{y}{x}=\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}$

∴ $(1-u)\left\{\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}\right\}=y^2$ ∴ $(1-u)\frac{1}{x}\frac{du}{dx}-\frac{u(1-u)}{x^2}=y^2$

∴ $x(1-u)\frac{du}{dx}-u(1-u)=u^2$ ∴ $x(1-u)\frac{du}{dx}=u$ ∴ $\left(\frac{1}{u}-1\right)du=\frac{dx}{x}$

∴ $\log u-u=\log x+c$ ∴ $\log\frac{u}{x}=u+c$ ∴ $\log y=xy+c$ ∴ $y=e^{xy+c}=e^c\cdot e^{xy}$

∴ $y=ce^{xy}$

(4) $u=x-y$ とおくと $\frac{du}{dx}=1-y'$ ∴ $-\frac{du}{dx}=u\tan x$ ∴ $\frac{du}{u}=-\tan x\,dx$

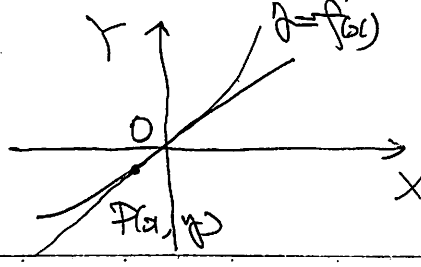
∴ $\frac{du}{u}=\frac{-\sin x}{\cos x}dx$ ∴ $\log u=\log\cos x+c=\log e^c\cdot\cos x$ ∴ $u=e^c\cdot\cos x$

∴ $u=c\cdot\cos x$ ∴ $x-y=c\cos x$ ∴ $y=x-c\cos x$

$\boxed{3}$ $u=ax+by+c$ とおくと $\frac{du}{dx}=a+by'$ ∴ $y'=\frac{1}{b}\left(\frac{du}{dx}-a\right)$

∴ $\frac{1}{b}\left(\frac{du}{dx}-a\right)=f(u)$ ∴ $\frac{du}{dx}=bf(u)+a$ ← xとuについて変数分離形

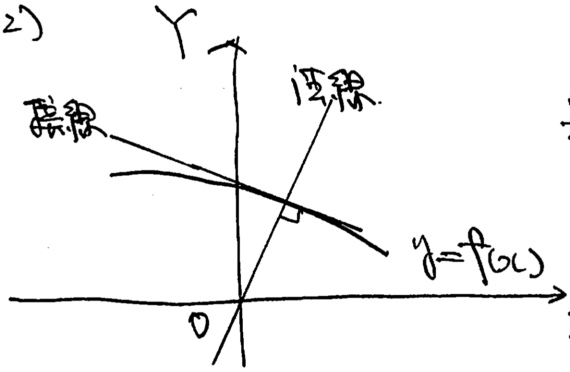
$\boxed{4}$ (1) 求める曲線を $y=f(x)$ とする 接線の方程式: $Y-y=\frac{dy}{dx}(X-x)$



原点を通り $0-y=\frac{dy}{dx}(0-x)$ ∴ $x\frac{dy}{dx}=y$ ∴ $\frac{dy}{y}=\frac{dx}{x}$

∴ $\log y=\log x+c=\log e^c\cdot x$ ∴ $y=e^c\cdot x$ ∴ $y=cx$

(2)



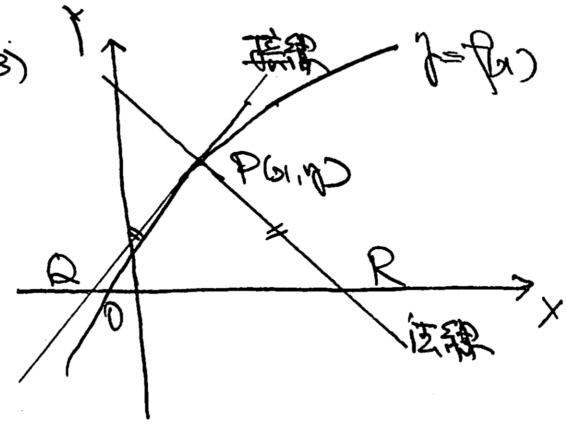
法線の方程式 $Y-y=-\dfrac{1}{\frac{dy}{dx}}(X-x)$

題意より $0-y=-\dfrac{1}{\frac{dy}{dx}}(0-x)$ $\therefore y\dfrac{dy}{dx}=-x$

$\therefore y\,dy=-x\,dx$ $\therefore \dfrac{y^2}{2}=-\dfrac{x^2}{2}+C$

$\therefore x^2+y^2=2C$ $\therefore \underline{x^2+y^2=C}$

(3)



接線の方程式 : $Y-y=\dfrac{dy}{dx}(X-x)$

Qの座標 : $\left(x-\dfrac{y}{\frac{dy}{dx}},\ 0\right)$

法線の方程式 : $Y-y=-\dfrac{1}{\frac{dy}{dx}}(X-x)$

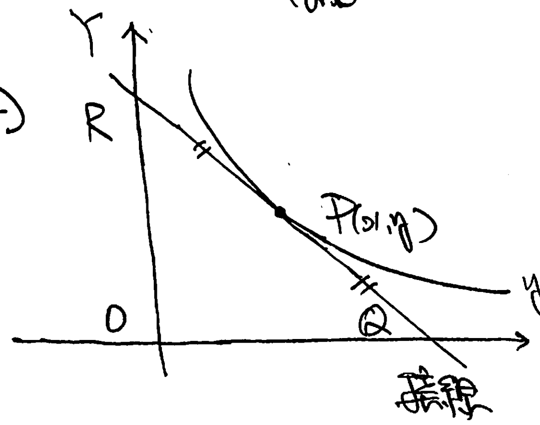
Rの座標 $\left(x+\dfrac{dy}{dx}y,\ 0\right)$

$\therefore \overline{PQ}^2=\left\{x-\left(x-\dfrac{y}{\frac{dy}{dx}}\right)\right\}^2+y^2=\dfrac{y^2}{\left(\frac{dy}{dx}\right)^2}+y^2$

$\overline{PR}^2=\left\{x-\left(x+\dfrac{dy}{dx}\cdot y\right)\right\}^2+y^2=\left(\dfrac{dy}{dx}\right)^2y^2+y^2$

$\overline{PQ}^2=\overline{PR}^2$ より $\dfrac{y^2}{\left(\frac{dy}{dx}\right)^2}=\left(\dfrac{dy}{dx}\right)^2y^2$ $\therefore \left(\dfrac{dy}{dx}\right)^4=1$ $\therefore \dfrac{dy}{dx}=\pm1$ $\therefore \underline{y=\pm x+C}$

(4)



接線の方程式 $Y-y=\dfrac{dy}{dx}(X-x)$

Qの座標 $\left(x-\dfrac{y}{\frac{dy}{dx}},0\right)$, Rの座標 $\left(0,\ y-x\dfrac{dy}{dx}\right)$

題意より $x=\dfrac{x-\frac{y}{\frac{dy}{dx}}}{2}$ かつ $y=\dfrac{y-x\frac{dy}{dx}}{2}$

$\therefore y=-x\dfrac{dy}{dx}$ $\therefore \dfrac{dy}{y}=-\dfrac{dx}{x}$

$\therefore \log y=-\log x+C=\log\dfrac{e^C}{x}$ $\therefore xy=e^C$ $\therefore \underline{xy=C}$