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Coalition-proof Nash Equilibria in a Normal-form Game and its Subgames

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Coalition-proof Nash Equilibria in a Normal-form Game and its Subgames

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Abstract

The relationship between coalition-proof equilibria in a normal-form game and those in its subgame is examined. A subgame of a normal-form game is a game in which the strategy sets of all players in the subgame are subsets of those in the normal-form game. Ray (2001) proved that a Nash equilibrium of a subgame is a Nash equilibrium of the original game under the condition of *no unilateral benefit*. Ray (2001) also established that a strong equilibrium (Aumann, 1959) of a subgame is a strong equilibrium of the original game under the condition of *no coalitional benefit*. However, under the condition of no coalitional benefit, the coalition-proof equilibria (Bernheim et al., 1987) of a subgame are not necessarily those of the original game. In this paper, we show that every coalition-proof equilibrium in a subgame is that in the original game if every strategy set is a subset of the real line, the condition of no unilateral benefit holds, and the original game satisfies *monotone externality*, *anonymity*, and *strategic substitutability*.

JEL classificaion: C72.

Key Words: Coalition-proof equilibrium; No unilateral benefit; Anonymity; Monotone externality; Strategic substitutability.

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1 Introduction

In this paper, we consider the relationship between coalition-proof equilibria in a normal-form game and those in its subgame. A subgame of a normal-form game is a restricted game in which the strategy sets for all players in the subgame are subsets of those in the normal-form game. Gilboa et al. (1990) called such a restricted game a subgame.

A Nash equilibrium of a subgame is not necessarily a Nash equilibrium of the original game. Ray (2001) studied sufficient conditions under which a Nash equilibrium of a subgame is that of the original game. He showed that, if the games satisfy the condition of *no unilateral benefit* (NUB), then a Nash equilibrium of a subgame is also a Nash equilibrium in the original normal-form game. The condition of NUB requires that no players can achieve a better payoff by playing a strategy outside the subgame, keeping the strategy of the others fixed. Ray (2001) also reported that every strong equilibrium in a subgame is also a strong equilibrium in the original game under the condition of *no coalitional benefit* (NCB), which requires that no group of players can be better off by playing strategy profiles outside the subgame.

However, surprisingly, coalition-proof equilibria of a subgame are not necessarily those in the original game even if the original game and its subgame satisfy no coalitional benefit. Ray (2001) provided an example in which a two-person normal-form game and its subgame satisfy NCB, the set of coalition-proof equilibria in the subgame and that in the original game are both non-empty, and their intersection is empty. Ray (2001) conjectured that the sufficient conditions would be very strong due to the recursive nature of coalition-proofness, and he pointed out the difficulty of establishing such sufficient conditions.

In this paper, we provide a sufficient condition under which every coalition-proof equilibrium in a subgame is also coalition-proof in the original game. We focus on normal-form games, in which strategy sets of all players are subsets of the real line. The sufficient conditions are: (i) the original game and its subgame satisfy NUB, and (ii) the original game satisfies the condition of *anonymity*, that of *monotone externality*, and that of *strategic substitutability*. Anonymity states that the payoffs of every player depend on his strategy and on the sum of the strategy of others. Monotone externality requires that a switch in a player's strategy change the payoffs to all the other players in the same direction. Strategic substitutability means that the incentive to every player to reduce his strategy becomes higher as the sum of the strategy of the other players increases. The conditions of anonymity, monotone externality, and strategic substitutability are satisfied in many normal-form games that have interested economists, such as the standard Cournot oligopoly game and voluntary contribution games to public goods. Finally, we apply our result to the Cournot oligopoly game with capacity constraints.

2 The Model

We consider two normal-form games, $\Gamma_1 = [N, (S_{1i})_{i\in N}, (u_{1i})_{i\in N}]$ and $\Gamma_2 = [N, (S_{2i})_{i\in N}, (u_{2i})_{i\in N}]$. For the game Γ_k ($k = 1, 2$), $N = \{1, 2, \ldots, n\}$ is the set of players, S_{ki} is a strategy set of player $i \in N$, and $u_{ki} : \prod_{j\in N} S_{kj} \to \mathbb{R}$ is a payoff function for player i. For all coalitions $J \subseteq N$, the complement of J is denoted by $-J$. Let us denote $\prod_{i\in J} S_{ki}$ by S_{kJ}. For notational convenience, we denote $\prod_{j\in N} S_{kj}$ by S_k for every $k \in \{1, 2\}$.

Definition 1 (Gilboa, et al., 1990) A normal-form game Γ_2 is a *subgame* of Γ_1 if the strategy set of all players in Γ_2 is a subset of his strategy set in Γ_1: $S_{2i} \subseteq S_{1i}$ for every $i \in N$, and $u_{2i}(s) = u_{1i}(s)$ for every i and for every $s \in S_1 \cap S_2$.

If Γ_2 is a subgame of Γ_1, the sets of strategies for all players in Γ_2 are subsets of those in Γ_1, and payoffs to all players in Γ_2 are equal to those in Γ_1 at corresponding strategy profiles.

Definition 2 (No Unilateral Benefit (NUB)) Let Γ_2 denote a subgame of Γ_1. The games satisfy NUB if, for every $i \in N$, for every $t_{1i} \in S_{1i} \setminus S_{2i}$, and for every $s_{2\,N\setminus\{i\}} \in S_{2\,N\setminus\{i\}}$, there is $t_{2i} \in S_{2i}$ such that $u_{2i}(t_{2i}, s_{2\,N\setminus\{i\}}) \geq u_{1i}(t_{1i}, s_{2\,N\setminus\{i\}})$.

The condition of NUB requires that no players can achieve a better payoff by playing a strategy outside the subgame, keeping the strategies of others fixed.

Definition 3 (No Coalitional Benefit (NCB)) Let Γ_2 denote a subgame of Γ_1. The games satisfy NCB if, for every $J \subseteq N$, for every $t_{1J} \in S_{1J} \setminus S_{2J}$, and for every $s_{2N\setminus J} \in S_{2N\setminus J}$, there exists $t_{2J} \in S_{2J}$ such that $u_{2i}(t_{2J}, s_{2N\setminus J}) \geq u_{1i}(t_{1J}, s_{2N\setminus J})$ for all $i \in J$.

Condition NCB requires that no group of players can be better off by playing strategy profiles outside the subgame. Clearly, if Γ_1 and Γ_2 satisfy NCB, then these games satisfy NUB.

In the following, the equilibrium concepts of a normal-form game are introduced. The first is the notion of *strong equilibrium* introduced by Aumann (1959).

Definition 4 (Strong equilibrium) A strategy profile $s^* \in S_k$ is a strong equilibrium of Γ_k if there exist no coalition $J \subseteq N$ and no strategy profile $\widetilde{s}_J \in S_{kJ}$ such that $u_{ki}(\widetilde{s}_J, s^*_{-J}) > u_{ki}(s^*)$ for all $i \in J$.

A strong equilibrium is a strategy profile that is immune to all possible coalitional deviations. Thus, a strong equilibrium is a Nash equilibrium, but the converse is not necessarily true.

The second notion is a *coalition-proof equilibrium*, which was introduced by Bernheim et al. (1987). Before introducing the notion of coalition-proof equilibria, we need to present a *restricted game*. For every normal-form game Γ_k, a *restricted game* with respect to a strategy profile $s \in S_k$ and a coalition $J \subseteq N$ denotes the game induced on the coalition J by strategies s_{-J}: $\Gamma_k^{J,s} = [J, (S_{ki})_{i \in J}, (u'_{ki})_{i \in J}]$, where $u'_i : \prod_{j \in J} S_{kj} \to \mathbb{R}$ is given by $u'_{ki}(t_J) = u_{ki}(t_J, s_{-J})$ for all $i \in J$ and $t_J \in \prod_{j \in J} S_{kJ}$.

Definition 5 A *coalition-proof equilibrium* $(s^*_1, \ldots, s^*_n)$ is defined inductively with respect to the number of players t:

- When $t = 1$, for all $i \in N$, s^*_i is a coalition-proof equilibrium for $\Gamma_k^{\{i\},s^*}$ if $s^*_i \in \arg\max \; u_{ki}(s_i, s^*_{-i})$ s.t. $s_i \in S_{ki}$.
- Let $T \subseteq N$ with $t = \#T \geq 2$. Assume that coalition-proof equilibria have been defined for all normal-form games with fewer players than t. Consider the restricted game Γ_k^{T,s^*} with t players.
  - A strategy profile $s^*_T \in S_{kT}$ is called *self-enforcing* if, for all $J \subsetneq T$, s^*_J is a coalition-proof equilibrium of Γ_k^{J,s^*}.
  - A strategy profile s^*_T is a coalition-proof equilibrium of Γ_k^{T,s^*} if it is a self-enforcing strategy profile and there is no other self-enforcing strategy profile $\widehat{s}_T \in S^{kT}$ such that $u_{ki}(\widehat{s}_T, s^*_{-T}) > u_{ki}(s^*_T, s^*_{-T})$ for all $i \in T$.

A coalition-proof equilibrium is clearly a Nash equilibrium in every normal-form game. Since a coalition-proof equilibrium is stable only against *self-enforcing* coalitional deviations, the set of coalition-proof equilibria contains that of strong equilibria.

Proposition 1 (Ray, 2001) Let Γ_2 denote a subgame of Γ_1. (i) Any Nash equilibrium of Γ_2 is a Nash equilibrium of Γ_1 if NUB holds. (ii) Every strong equilibrium in Γ_2 is a strong equilibrium in Γ_1 if NCB holds.

Example 1 indicates that a coalition-proof equilibrium in a subgame is not necessarily coalition-proof in the original game under the condition of NCB.

Example 1 (Ray, 2001) Consider the two-player games in Tables 1 and 2. In the two normal-form games, player 1 chooses rows, and player 2 chooses columns. A vector in each cell represents a vector of payoffs, in which the first entry is player 1's payoff and the second entry is player 2's payoff. Note that Γ_2 is a subgame of Γ_1, and Γ_1 and Γ_2 satisfy NCB. In these games, a profile of strategies (B, L) is a Nash equilibrium. However, (B, L) is coalition-proof in Γ_2, while (B, L) is not coalition-proof in Γ_1.

3 Results

In this section, we consider a class of games with n players in which the strategy space of each player is a subset of the real line: for each game Γ_k ($k = 1, 2$) and for each $i \in N$, $S_{ki} \subseteq \mathbb{R}$. We introduce a condition of *anonymity*, that of *monotone externality*, and that of *strategic substitutability.*

The anonymity condition means that the payoff function of every player depends on his strategy and on the aggregate strategy of all other players.

Definition 6 (Anonymity) A game Γ_k satisfies *anonymity* condition if the following condition holds: for all $i \in N$, all $s_i \in S_{ki}$, and all s_{-i}, $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j \neq i} s_j = \sum_{j \neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) = u_{ki}(s_i, \widehat{s}_{-i})$.

The next condition is that of *monotone externality.* The condition states that the payoffs to every player are either non-increasing or non-decreasing with respect to the sum of strategies of the other players.

Definition 7 (Monotone externality) The game Γ_k satisfies the condition of *monotone externality* if the game satisfies either (i) or (ii).

(i) The game Γ_k satisfies the condition of *positive externality* if the game satisfies the following condition: for all $i \in N$, all $s_i \in S_{ki}$, and all s_{-i} and $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j \neq i} s_j > \sum_{j \neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) \geq u_{ki}(s_i, \widehat{s}_{-i})$ holds.

(ii) The game Γ_k satisfies the condition of *negative externality* if the game satisfies the following condition: for all $i \in N$, all $s_i \in S_{ki}$, and all s_{-i} and $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j \neq i} s_j > \sum_{j \neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) \leq u_{ki}(s_i, \widehat{s}_{-i})$ holds.

The third condition is that of *strategic substitutability.* Under this condition, the incentive of every player to reduce his strategy becomes higher as the sum of the strategies of other players increases.

Definition 8 (Strategic substitutability) The game Γ_k satisfies the condition of *strategic substitutability* if the following condition holds: for all $i \in N$, all s_i, $\widehat{s}_i \in S_{ki}$, and all s_{-i}, $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $s_i > \widehat{s}_i$ and $\sum_{j \neq i} s_j > \sum_{j \neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) - u_{ki}(\widehat{s}_i, s_{-i}) < u_{ki}(s_i, \widehat{s}_{-i}) - u_{ki}(\widehat{s}_i, \widehat{s}_{-i})$.

Let Γ_1 denote a normal-form game, and let Γ_2 denote a subgame of Γ_1. It is worth noting that Γ_2 satisfies the three conditions if Γ_1 does. The following proposition provides a sufficient condition under which the set of coalition-proof equilibria in Γ_2 is included in the set of coalition-proof equilibria in Γ_1.

Proposition 2 Let Γ_1 be a normal-form game, and let Γ_2 be a subgame of Γ_1. Suppose that the sets of strategies for all players in Γ_1 consist only of real numbers. If Γ_1 and Γ_2 satisfy NUB and Γ_1 satisfies anonymity, monotone externality, and strategic substitutability, then every coalition-proof equilibrium in Γ_2 is also coalition-proof in Γ_1.

Proof. Let $s^* \in S_2$ denote a coalition-proof equilibrium in Γ_2. We prove that s^* is coalition-proof in Γ_1. Let us suppose, to the contrary, that s^* is not coalition-proof in Γ_1. Then, there is a coalition $J \subseteq N$ and its strategy profile $t_J \in S_{1J}$ such that t_J is coalition-proof in $\Gamma_1^{J,\,s^*}$ and $u_{1i}(s^*_J, s^*_{N\setminus J}) < u_{1i}(t_J, s^*_{N\setminus J})$ for every $i \in J$. Suppose that Γ_1 and Γ_2 satisfy NUB and Γ_1 satisfies anonymity, positive externality, and strategic substitutability.*1 Note that s^* is also a Nash equilibrium in Γ_1 by NUB.*2 Note also that $t_J \notin S_{2J}$.

*1 We can similarly prove the statement in the case of negative externality.

*2 It follows from this that $\#J \geq 2$.

We first prove the following lemma.

Lemma 1 It follows that $\sum_{j \in J \setminus \{i\}} t_j > \sum_{j \in J \setminus \{i\}} s_j^*$ for every $i \in J$.

Proof of Lemma 1. Suppose, to the contrary, that $i \in J$ exists such that $\sum_{j \in J \setminus \{i\}} t_j \leq \sum_{j \in J \setminus \{i\}} s_j^*$. By the definition of Nash equilibrium, we have $u_{1i}(s_J^*, s_{-J}^*) \geq u_{1i}(t_i, s_{J \setminus \{i\}}^*, s_{-J}^*)$. By the condition of positive externality, we obtain $u_{1i}(t_i, t_{J \setminus \{i\}}, s_{-J}^*) \geq u_{1i}(t_i, t_{J \setminus \{i\}}, s_{-J}^*)$. Therefore, we have $u_{1i}(s_J^*, s_{-J}^*) \geq u_{1i}(t_i, t_{J \setminus \{i\}}, s_{-J}^*)$, which is a contradiction. **(End of Proof of Lemma 1)**

Lemma 2 Profile t_J is not a Nash equilibrium of Γ_1^{J,s^*}.

Proof of Lemma 2. It follows from Lemma 1 that $\sum_{i \in J} \sum_{j \in J \setminus \{i\}} t_j > \sum_{i \in J} \sum_{j \in J \setminus \{i\}} s_j^*$. Thus, we have $\sum_{i \in J} t_i > \sum_{i \in J} s_i^*$, which implies that there is $j \in J$ such that $t_j > s_j^*$. By the definition of a Nash equilibrium, we have $u_{1j}(s_j^*, s_{-j}^*) - u_{1j}(t_j, s_{-j}^*) \geq 0$. From the condition of strategic substitutability, we obtain $u_{1j}(s_j^*, t_{J \setminus \{i\}}, s_{-J}^*) - u_{1j}(t_j, t_{J \setminus \{i\}}, s_{-J}^*) > u_{1j}(s_j^*, s_{-j}^*) - u_{1j}(t_j, s_{-j}^*)$. As a result, we have $u_{1j}(s_j^*, t_{J \setminus \{i\}}, s_{-J}^*) > u_{1j}(t_j, t_{J \setminus \{i\}}, s_{-J}^*)$. Therefore, t_J is not a Nash equilibrium of Γ_1^{J,s^*}. **(End of Proof of Lemma 2)**

Since every coalition-proof equilibrium is a Nash equilibrium, t_J is not coalition-proof in Γ_1^{J,s^*}. This is a contradiction. Therefore, s^* is coalition-proof in Γ_1. ■

The following are remarks concerning the main result.

Remark 1 Without one of the three conditions, a coalition-proof Nash equilibrium in a subgame may not be a coalition-proof equilibrium in the original game. In Example 2, the original game and its subgame satisfy NUB, and the original game satisfies monotone externality but not strategic substitutability. In Example 3, the original game and its subgame satisfy NUB, and the original game satisfies strategic substitutability but not monotone externality. In both examples, a coalition-proof equilibrium in a subgame is not that in the original game. Therefore, the three conditions play an important role in establishing the main result.

Example 2 Consider the two-player games, Γ_1 and Γ_2, which are shown in Tables 3 and 4, respectively. Note that these two games satisfy NCB. Note also that Γ_1

satisfies the conditions of anonymity and positive externality. However, the condition of strategic substitutability does not hold. For example, if strategic substitutability holds, then $u_{11}(1,1)-u_{11}(3,1) < u_{11}(1,2)-u_{11}(3,2) < u_{11}(1,3)-u_{11}(3,3)$; however, we obtain $u_{11}(1,1)-u_{11}(3,1) = 20$, $u_{11}(1,2)-u_{11}(3,2) = 0$, and $u_{11}(1,3)-u_{11}(3,3) = 10$ in Γ_1. The only coalition-proof equilibrium is (1, 1) in Γ_2, while (2, 2) is the only coalition-proof equilibrium in Γ_1. Thus, a coalition-proof equilibrium in Γ_2 is not coalition-proof in Γ_1.

Example 3 Consider the two-player games, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, which are depicted in Tables 5 and 6, respectively. Note that $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ satisfy NCB and $\widetilde{\Gamma}_1$ satisfies the strategic substitutability condition. However, $\widetilde{\Gamma}_1$ does not satisfy the condition of monotone externality. If the monotone externality condition holds, then either $u_{11}(2,1) \leq u_{11}(2,2) \leq u_{11}(2,3)$ or $u_{11}(2,1) \geq u_{11}(2,2) \geq u_{11}(2,3)$ holds; however, we have $u_{11}(2,1) = 0$, $u_{11}(2,2) = 40$, and $u_{11}(2,3) = 30$. Clearly, profile (3,1), which is coalition-proof in $\widetilde{\Gamma}_2$, is not a coalition-proof equilibrium in $\widetilde{\Gamma}_1$.

Remark 2 The condition of monotone externality and that of strategic substitutability cannot be dropped even if a normal-form game and its subgame satisfy NCB. This is demonstrated in Example 2 and Example 3. In these examples, the original game and its subgame satisfy NCB, which is stronger than NUB. Nevertheless, the set of coalition-proof equilibria in the original game and that in the subgame are disjointed.

The conditions of anonymity, monotone externality, and strategic substitutability are satisfied in many economic games, such as the standard Cournot oligopoly game and voluntary contribution games to public goods. In the following example, we demonstrate that the standard Cournot oligopoly game has a subgame such that the Cournot game and the subgame satisfy NUB and every coalition-proof equilibrium of the subgame is also a coalition-proof equilibrium of the Cournot game.

Example 4 (Cournot Competition under Capacity Constraints) Consider a Cournot competition game by two firms that produce a homogeneous good. The inverse demand function of the good is given by $P(Q) = \max\{a - Q, 0\}$, in which a is positive and Q is the sum of outputs of the two firms. Let us assume that c is the marginal cost of producing the good and there is no fixed cost. We assume that $a > c > 0$.

We consider two Cournot competition games, Γ_1 and Γ_2. Every firm has no capacity constraint in Γ_1. On the other hand, in Γ_2, each firm faces the capacity constraint and can produce the good up to $a - c$ units. Then, Γ_2 is a subgame of Γ_1. The best response graphs in each game are shown in Figure 1 and Figure 2. In each figure, q_1 and q_2 are the outputs of firms 1 and 2, respectively. The red line represents the best response function of firm 1, and the blue one represents the best response function of firm 2. Point E_1 is a Nash equilibrium in Γ_1, and E_2 is a Nash equilibrium in Γ_2. Note that $(q_1, q_2) = (\frac{a-c}{3}, \frac{a-c}{3})$ in E_1 and E_2. In this example, Γ_1 and Γ_2 satisfy NUB, and Γ_1 satisfies anonymity, negative externality, and strategic substitutability.

In both games, only the profile of outputs $(q_1, q_2) = (\frac{a-c}{3}, \frac{a-c}{3})$ is a Nash equilibrium, and the profile is coalition-proof. Thus, the sets of coalition-proof equilibria in Γ_1 and Γ_2 coincide.

4 Conclusion

In this paper, the relationship between coalition-proof equilibria of a normal-form game and its subgame was examined. Ray (2001) showed that a Nash equilibrium in a subgame is that in the original game under the condition of NUB and a strong equilibrium of a subgame is that of the original game under NCB. However, a coalition-proof equilibrium of a subgame is not necessarily coalition-proof in the original game even if NCB holds, and Ray (2001) pointed out the difficulty of providing a sufficient condition under which the set of coalition-proof equilibria of the original game and that of its subgame are related by inclusion. However, in this paper, we provided such a sufficient condition, which is satisfied in games studied by economists. We focused on a class of games in which sets of strategies for all players consist only of the real numbers and proved that every coalition-proof equilibrium of a subgame is also coalition-proof in the original game if the original game and the subgame satisfy NUB and the original game satisfies the condition of anonymity, that of monotone externality, and that of strategic substitutability.

The coalition-proof equilibrium is well known as a refinement of the Nash equilibrium. However, little is known about the structure of the equilibrium. This paper has studied coalition-proof equilibria of a normal-form game and its subgames. Clarifying other properties of this equilibrium concept will be left for future studies.

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1 \ 2	L	R	Y
X	0, 0	0, 0	3, 3
T	0, 5	4, 4	0, 0
B	2, 2	5, 0	0, 0

Table. 1 The originial game Γ_1

1 \ 2	L	R
T	0, 5	4, 4
B	2, 2	5, 0

Table. 2 A subgame Γ_2 of Γ_1

1 \ 2	1	2	3
1	20, 20	20, 20	50, 0
2	20, 20	30, 30	35, 20
3	0, 50	20, 35	40, 40

Table. 3 A two-player game Γ_1

1 \ 2	1	3
1	20, 20	50, 0
3	0, 50	40, 40

Table. 4 A subgame Γ_2 of Γ_1

1 \ 2	1	2	3
1	-30, 30	30, 30	40, 40
2	0, 50	40, 40	30, 30
3	20, 20	50, 0	30, -30

Table. 5 A two-player game $\widetilde{\Gamma}_1$

1 \ 2	1	2
2	0, 50	40, 40
3	20, 20	50, 0

Table. 6 A subgame $\widetilde{\Gamma}_2$ of $\widetilde{\Gamma}_1$

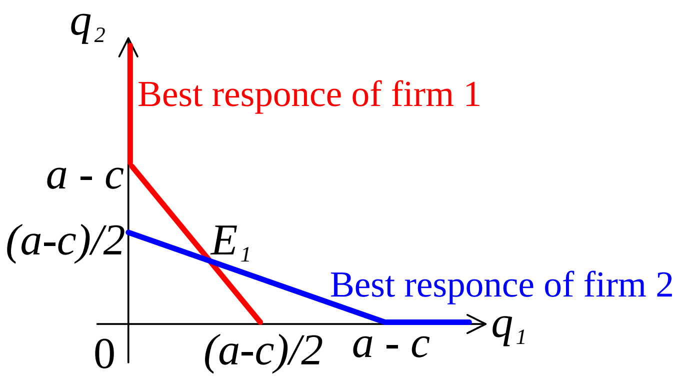

Fig. 1 The best-responce graph in Γ_1

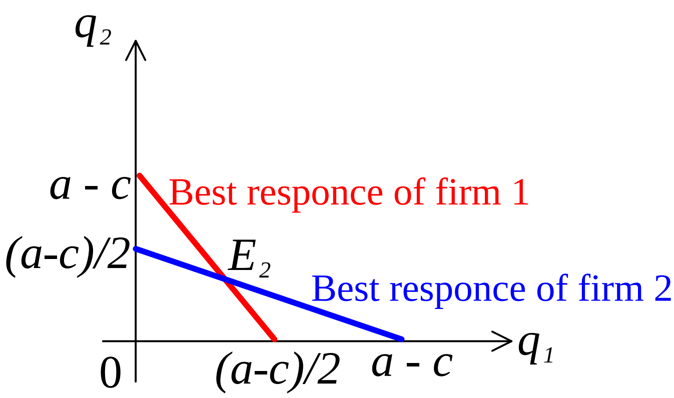

Fig. 2 The best-responce graph in Γ_2