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The Possibility of Efficient Provision of a Public Good in Voluntary Participation Games

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Abstract

In this study, we examine the allocative efficiency of Nash equilibria in a voluntary participation game in which a public good is provided in units of non-negative integers. We show that the participation game has a Nash equilibrium that supports an efficient allocation and that some Nash equilibria are strong equilibria if at most one unit of the public good can be provided. However, the Nash equilibria of the participation game do not necessarily support efficient allocations if up to two units of the public good can be provided. We investigate the possibility of attaining efficient allocations at Nash equilibria in the case in which at most two units of the public good can be produced. We prove that Nash equilibria are less likely to support efficient allocations if the participation of many agents is needed for the efficient provision of the public good in the case of identical agents.

Keywords: Participation game, Nash equilibrium, Efficiency, Public project, Multi-unit public good, Diminishing rate of marginal benefits.

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1 Introduction

This paper is a presentation of a participation problem in a mechanism to produce a pure public good. From the theory of implementation, the construction of a mechanism can solve the "free-rider" problem in economies with public goods. For example, Bagnoli and Lipman (1989), Jackson and Moulin (1992), and Bag (1997) constructed mechanisms to implement desirable allocation rules in an economy with a discrete public good.

However, the implementation theory assumes the participation of all agents, and each agent lacks the right to determine whether or not to participate in the mechanism. Palfrey and Rosenthal (1984), Saijo and Yamato (1999), and Dixit and Olson (2000) pointed out the importance of strategic behavior of agents as they decide whether or not to participate in the mechanisms. In the real world, for example, in the participation problems in international environmental treaties, agents often have the right to make such decisions, and they may have an incentive not to enter the mechanism, hoping that other agents will participate in the mechanism and provide a public good. This will generate another kind of a free-rider problem.

These authors formulated a participation game in a public good mechanism. In the game, each agent simultaneously chooses whether or not to participate in the mechanism. If an agent chooses to participate, he pays the expense requested by the mechanism, and the public good is produced. If an agent chooses not to participate, he can enjoy the public good at no cost. Palfrey and Rosenthal (1984) and Dixit and Olson (2000) analyzed the participation game in a case in which the public good is discrete and at most one unit of public good is produced. They showed that there are pure-strategy Nash equilibria that support efficient allocations in the game. Saijo and Yamato (1999) examined the participation game with a perfectly divisible public good. They considered a mechanism that implements the Lindahl allocation rule. They showed that not every agent enters the mechanism at pure-strategy Nash equilibria and proved that efficient allocations of the economy are not achieved at the equilibrium of the game in many cases. Hence, it depends on the type of provision of the public good whether or not the equilibrium of the participation game achieves efficient allocations.

In this paper, we examine a participation game that is similar to a model presented in earlier literature. We consider an economy in which the public good is discrete and in which there is a mechanism that implements a Pareto efficient and individually rational allocation rule. First, we examine a participation game in which only one unit of the public good can be provided, which is similar to Palfrey and Rosenthal (1984) and Dixit and Olson (2000) (hereafter, this game is called a *participation game with a public project*). We investigate the equilibria of the participation game in a mechanism that implements a proportional cost-sharing rule: the public good is produced in a way that maximizes the total surplus of participants, and the cost of producing the public good is distributed among participants in proportion to the benefits that participants receive from the public good. In this game, there is a pure-strategy Nash equilibrium at which a Pareto efficient allocation is achieved, and some of such Nash equilibria are the strong equilibria introduced by Aumann (1959).

Secondly, we extend our analysis to the case of a multi-unit public good. In particular, we focus on a participation game in which the public good is provided in units of non-negative integers and only up to two units can be provided. In this case, the Nash equilibria of the game do not necessarily support the efficient allocation. We characterize Nash-equilibrium sets of participants, and we examine how possible it is that efficient allocations are attained at Nash equilibria in the case of identical agents. We characterize the set of preference parameters in which the efficient allocations are supportable as Nash equilibria by using a diminishing rate of marginal benefits. We characterize the range of diminishing rates of marginal benefits in which the efficient allocations are supported as Nash equilibria. We prove that the range of diminishing rates of marginal benefits shrinks as the number of participants increases. From this result, the efficient provision can be achieved at few preference parameters if the participation of many agents is necessary for the efficient provision of the public good. Therefore, we can say that the efficient provision of the public good is rarely achieved if the cooperation of many agents is necessary for the efficient provision. It also follows from this result that the inefficient provision of the public good arises even in the participation game in which only up to two units of the public good can be provided; this phenomenon is similar to that presented by Saijo and Yamato (1999) in the case of a perfectly divisible public good.

Before the model is introduced, the relationship between this work and others will be discussed. First, in the participation game with a public project, our model allows the possibility that agents have different preferences and the participants share the cost of the project according to the proportional cost-sharing rule, while agents are identical and the participants share the cost evenly in the models of Palfrey and Rosenthal (1984) and Dixit and Olson (2000). The introduction of the proportional cost-sharing rule when agents are heterogeneous would be a natural extension of models presented in earlier literature. Since the proportional cost-sharing rule can be implemented by mechanisms of Jackson and Moulin (1992), the participation problem with a public project in this paper can be interpreted as a participation game in the mechanisms of Jackson and Moulin (1992).

Secondly, we consider the possibility that agents form a coalition and coordinate the participation decisions. Palfrey and Rosenthal (1984), Saijo and Yamato (1999), and Dixit and Olson (2000) have focused solely on Nash equilibria, disregarding the effects. However, there are real-world examples in which agents negotiate the participation decisions. In the case of the Kyoto Protocol, the ratification of Russia was essential to bring the protocol into force. The European Union, which is an environmentally conscious group, negotiated with Russia and tried to induce Russia to ratify the protocol. As a reduced form of such a bargaining process, we investigate strong equilibria (Aumann, 1959) of the participation game. A strong equilibrium is a strategy profile that is stable against all possible coalitional deviations. This is a very demanding equilibrium concept, and many games that are of interest to economists do not have a strong equilibrium. In this paper, we provide a sufficient condition for the existence of a strong equilibrium in games of the provision of pure public good.

Third, we extend the models of earlier literature to a participation game with a discrete and multi-unit public good. Palfrey and Rosenthal (1984), Dixit and Olson (2000), and Shinohara (2004) examined participation games with a public project, and these authors proved that the games have Nash equilibria to produce efficient allocations. However, few studies have investigated whether or not there is a Nash equilibrium that supports an efficient allocation in the participation game when the public good is discrete and multi-unit. In this paper, we try to clarify the characteristic of Nash equilibria when the public good is provided in multiple units and at most two units of the public good are produced.

2 A participation game with a public project

We consider the problem of undertaking a (pure) public project and distributing its cost. Let n be the number of agents. We denote the set of agents by $N = \{1, \dots, n\}$. Let $y \in \{0, 1\}$ be the public project: $y = 1$ if the project is undertaken, and $y = 0$ if not. Let $\theta_i > 0$ denote agent i's *willingness to pay* for the project or *benefit* from the project. Let $x_i \geq 0$ denote a transfer from agent i. Each agent i has a preference relation which is represented by a quasi-linear utility function $V_i(y, x_i) = \theta_i y - x_i$.

The cost of the project is $c > 0$.

In this paper, we assume that there exists a mechanism that implements the proportional cost-sharing rule. We consider a two-stage game. In the first stage, each agent simultaneously decides whether she participates in the mechanism or not. In the second stage, following the rule of the mechanism, only the agents that selected participation in the first stage decide the implementation of the project and the distribution of its cost. As a result, the *proportional cost-sharing allocation* only for participants' preferences is achieved.

Let P be a set of participants and let $(y^P, (x_j^P)_{j\in N})$ be the outcome of the second stage when P is the set of participants. We denote $\theta_P = \sum_{j\in P}\theta_j$ for all sets of participants P: θ_P is the sum that agents in P are willing to pay for the public project. For all subsets P of N, $\#P$ means the cardinality of the set P.

Assumption 1 For every set of participants P, the allocation to the participants $(y^P, (x_j^P)_{j\in P})$ satisfies the following conditions:

$$\begin{aligned}
&\text{if } \theta_P > c, \text{ then } x_i^P = \frac{\theta_i}{\theta_P}c \text{ for all } i \in P \text{ and } y^P = 1, \text{ and} \\
&\text{if } \theta_P \leq c, \text{ then } x_i^P = 0 \text{ for all } i \in P \text{ and } y^P = 0.
\end{aligned}$$

The project is undertaken if and only if the sum that the participants are willing to pay for the project exceeds the project cost. If the project is undertaken, then the cost of the project is distributed among the participants in proportion to the the benefits that the participants receive from the project.

In this paper, we are not concerned with the implementation problem of the proportional cost-sharing rule. However, there are mechanisms in which the proportional cost-sharing rule is attained at equilibria. For example, Jackson and Moulin (1992) constructed a multi-stage mechanism which implements the proportional cost-sharing rule in subgame perfect Nash equilibria and undominated Nash equilibria. In these mechanisms, agents report an estimate of the collective benefit accruing from the project and their own benefit for the project. In equilibria of these mechanisms, agents truthfully announce the collective benefit and their own benefits.

Assumption 2 Let $P \subseteq N$ be a set of participants. We assume $x_i^P = 0$ for all $i \notin P$, and every non-participant can also consume y^P.

Assumption 2 expresses the non-excludability of the project. In this assumption, participants bear the cost share for the project, but non-participants do not. In spite of this, non-participants can benefit from the project.

Given the outcome of the second stage, the participation-decision stage can be reduced to the following simultaneous game. In the game, each agent i simultaneously chooses either $s_i = I$ (participation) or $s_i = O$ (non-participation), and then the set of participants is determined. Let P^s be the set of participants at an action profile $s = (s_1, \dots, s_n)$. Then each agent i obtains the utility $V_i(y^{P^s}, x_i^{P^s})$ at the action profile s. That is, if the public project is undertaken, then participants share the cost of it in proportion to the benefits from the project. Each non-participant can free-ride the public project. On the other hand, if the project is not carried out, then the payoffs of both participants and non-participants are zero. We call this reduced game *participation game* and formally define as follows.

Definition 1 (Participation game) A *participation game* is represented by $G = \left[N,\, S^n = \{I,\, O\}^n,\, (U_i)_{i\in N}\right]$, where U_i is the payoff function of i which associates a real number $U_i(s)$ with each strategy profile $s \in S^n$: if P^s designates the set of participants at s, then $U_i(s) = V_i(y^{P^s}, x_i^{P^s})$ for all i.

Our attention is limited to the pure strategy profiles.

The notions of equilibria of the participation game are defined as follows. The Nash equilibria of the participation game are defined as usual. First, a definition is given for a strict Nash equilibrium.

Definition 2 (Strict Nash equilibrium) A strategy profile $s^* \in S^n$ is a *strict Nash equilibrium* if, for all $i \in N$ and for all $\widehat{s}_i \in S \setminus \{s_i^*\}$, $U_i(s_i^*, s_{-i}^*) > U_i(\widehat{s}_i, s_{-i}^*)$.

Before defining strong equilibrium, some notation is presented. For all $D \subseteq N$, denote the complement of D by $-D$. For all coalitions D, $s_D \in S^{\#D}$ denotes a strategy profile for D. For all $s_N \in S^n$, denote s_N by s.

Definition 3 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a *strong equilibrium of* G if there exist no coalition $T \subseteq N$ and its strategy profile $\widetilde{s}_T \in S^{\#T}$ such that $\sum_{i\in T} U_i(\widetilde{s}_T, s_{-T}^*) > \sum_{i\in T} U_i(s^*)$ for all $i \in T$.

A strong equilibrium is a strategy profile at which no coalition, taking the strategies of others as given, can jointly deviate in a way that increases the sum of the payoffs of its members. The strong equilibrium in Definition 3 is slightly different from that originally defined by Aumann (1959). The difference lies in the possibility of monetary transfers among agents in coalitions. Our definition allows members of coalitions to freely send monetary transfers to each other, but Aumann (1959)'s definition does not. Hence, in our model, members of a coalition can coordinate their participation decision through monetary transfers. It is noteworthy that the set of strong equilibria in a game without monetary transfers generally contains the set of strong equilibria in the game with monetary transfers. However, the converse is not necessarily true. Obviously, all strict Nash equilibria and all strong equilibria are Nash equilibria. However, the set of strict Nash equilibria and that of strong equilibria are not always related by inclusion.

Example 1 Let $N = \{1, 2, 3\}$, $\theta_1 = \theta_2 = \theta_3 = 3/4$, and $c = 1$. The payoff matrix of this example is depicted in Table 1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria. One is the Nash equilibrium with two participants and the other is the Nash equilibrium with no participants. Only the Nash equilibria with participation of two agents are strict Nash equilibria and strong equilibria.

⟨Insert Table 1 here.⟩

3 Nash equilibria of the participation game

In this section, we characterize the sets of participants attained at Nash equilibria. The set of feasible allocations is defined as A:

$$A = \left\{ (y, (x_j)_{j \in N}) \mid x_j \geq 0 \text{ for all } j \in N,\ y \in \{0, 1\} \text{ and } \sum_{j \in N} x_j \geq cy \right\}.$$

Assumption 3 $\theta_N > c$.

Definition 4 An allocation $(y, (x_j)_{j \in N})$ is called *Pareto efficient* if there exists no feasible allocation $(\widehat{y}, (\widehat{x}_j)_{j \in N})$ such that $V_i(\widehat{y}, \widehat{x}_i) \geq V_i(y, x_i)$ for all $i \in N$ with strict inequality for at least one $i \in N$.

We, hereafter, consider a case in which Assumption 3 holds. By Assumption 3, the public project is undertaken at every Pareto efficient allocation. In the next Lemma, we characterize the sets of participants supported as Nash equilibria.

Lemma 1 (1.1) Let $P \subseteq N$ be such that $\theta_P > c$. Then, P is supported as a Nash equilibrium if and only if $\theta_P - \theta_i \leq c$ for all $i \in P$.

(1.2) Let $P \subseteq N$ be such that $\theta_P \leq c$. Then, P is supported as a Nash equilibrium if and only if $\theta_P + \theta_i \leq c$ for all $i \notin P$.

Proof. First, we show (1.1). Let P be a set of participants that satisfies $\theta_P > c$. Let $(y^P, (x_j^P)_{j \in P})$ denote the allocation when P is the set of participants.

Let us suppose that P is supported as a Nash equilibrium. Then, the following conditions are satisfied:

$$V_i(y^P, x_i^P) \geq V_i(y^{P\setminus\{i\}}, x^{P\setminus\{i\}}) \text{ for all } i \in P, \text{ and}$$
$$V_i(y^P, x_i^P) \geq V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}}) \text{ for all } i \notin P.$$

Since $\theta_P > c$, we have $V_i(y^P, x_i^P) = \theta_i - \dfrac{\theta_i}{\theta_P}c$ for all $i \in P$. If P is such that $\theta_P - \theta_j > c$ for some $j \in P$, then agent j has an incentive to switch from I to O because $V_j(y^{P\setminus\{j\}}, x_j^{P\setminus\{j\}}) = \theta_j > \theta_j - \dfrac{\theta_j}{\theta_P}c = V_j(y^P, x_j^P)$. This contradicts the assumption that P is attained at a Nash equilibrium. Therefore, we have $\theta_P - \theta_i \leq c$ for all $i \in P$. Conversely, suppose that $\theta_P - \theta_i \leq c$ for all $i \in P$. Then, we have

$$V_i(y^P, x_i^P) = \theta_i - \frac{\theta_i}{\theta_P}c \ > \ 0 = V_i(y^{P\setminus\{i\}}, x_i^{P\setminus\{i\}}) \text{ for all } i \in P, \text{ and}$$
$$V_i(y^P, x_i^P) = \theta_i \ > \ \theta_i - \frac{\theta_i}{\theta_P + \theta_i}c = V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}}) \text{ for all } i \notin P.$$

Hence, P is supportable as a Nash equilibrium.

Secondly, we prove (1.2). Let P be such that $\theta_P \leq c$. If $\theta_P + \theta_i \leq c$ for all $i \notin P$, then $V_i(y^P, x_i^P) = V_i(y^{P\setminus\{i\}}, x_i^{P\setminus\{i\}}) = 0$ for all $i \in P$ and $V_i(y^P, x_i^P) = V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}}) = 0$ for all $i \notin P$. Hence, P is attained at a Nash equilibrium. Conversely, suppose that P is supported as a Nash equilibrium. Then, we have $V_i(y^P, x_i^P) \geq V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}}) = 0$ for all $i \notin P$. Since $\theta_P \leq c$, we obtain $V_i(y^P, x_i^P) = 0$ for all $i \notin P$. If there exists agent $j \notin P$ such that $\theta_P + \theta_j > c$, then we obtain $V_j(y^{P\cup\{j\}}, x_j^{P\cup\{j\}}) = \theta_j - \dfrac{\theta_j}{(\theta_P + \theta_j)}c = \dfrac{\theta_j}{(\theta_P + \theta_j)}(\theta_P + \theta_j - c) > 0$. This means that agent j has an incentive to deviate, which is a contradiction. Therefore, it follows that $\theta_P + \theta_i \leq c$ for all $i \notin P$. ■

In the following Lemma, we show that there is a set of participants that satisfies (1.1) of Lemma 1.

Lemma 2 There exists a set of participants that satisfies (1.1) of Lemma 1 under Assumption 3 in the participation game. Therefore, there is a Nash equilibrium at which the project is carried out in the game.

Proof. Let P be a set of participants such that:

$$P \in \arg\min_{Q \subseteq N} \ \theta_Q \text{ such that } \theta_Q > c. \tag{1}$$

Note that there is at least one set of participants R satisfying $\theta_R > c$ by Assumption 3. If there is some $j \in P$ such that $\theta_P - \theta_j > c$, then $\theta_P > \theta_{P \setminus \{j\}} > c$. This contradicts (1), since θ_P is not the minimal number. Hence, P satisfies $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$. ■

Remark 1 The set of Nash equilibria in (1.1) of Lemma 1 coincides with that of strict Nash equilibria in the participation game. By Lemma 2, a strict Nash equilibrium exists in the participation game.

Nash equilibria in (1.2) of Lemma 1 are non-strict. Note that, if non-strict Nash equilibria exist, then the project is not done in the equilibrium, and the allocations supported as non-strict Nash equilibria are Pareto dominated by the allocations attained at strict Nash equilibria. The following proposition shows that the set of strict Nash equilibria coincides with the set of Nash equilibria that support efficient allocations.

Proposition 1 In the participation game, a strategy profile is a strict Nash equilibrium if and only if it is a Nash equilibrium at which an efficient allocation is attained.

Proof. First, we prove that every strict Nash equilibrium is a Nash equilibrium that supports an efficient allocation. Obviously, every strict Nash equilibrium is a Nash equilibrium. Hence, we need to show that every allocation achieved at a strict Nash equilibrium is Pareto efficient. Let $s \in S^n$ denote a strict Nash equilibrium and let P^s be the set of participants at s. Let us denote the allocation that is attained at s by $(y^{P^s}, (x_j^{P^s})_{j \in N})$. Note that $V_i(y^{P^s}, x_i^{P^s}) = \theta_i - \dfrac{\theta_i}{\theta_{P^s}} c$ for all $i \in P^s$ and $V_i(y^{P^s}, x_i^{P^s}) = \theta_i$ for all $i \notin P^s$. Suppose, on the contrary, that a feasible allocation $(\widehat{y}, (\widehat{x}_j)_{j \in N})$ Pareto dominates $(y^{P^s}, (x_j^{P^s})_{j \in N})$. It must be satisfied that $V_i(\widehat{y}, \widehat{x}_i) = \theta_i$ for all $i \notin P^s$ because θ_i is the greatest payoff of agent i in A. Hence, there is at least one participant $j \in P^s$ such that $V_j(\widehat{y}, \widehat{x}_j) > V_j(y^{P^s}, x_j^{P^s})$. Let $J \subseteq P^s$ be a set of such participants and let $\varepsilon_j = V_j(\widehat{y}, \widehat{x}_j) - V_j(y^{P^s}, x_j^{P^s}) > 0$ for all $j \in J$. Since $V_j(y^{P^s}, x_j^{P^s}) = \theta_j - \dfrac{\theta_j}{\theta_{P^s}} c > 0$ for every $j \in J$, we must have $\widehat{y} = 1$: otherwise, $V_j(\widehat{y}, \widehat{x}_i) = 0$. Then, we have $V_j(\widehat{y}, \widehat{x}_i) = \theta_j - x_j^P + \varepsilon_j$ for all $j \in J$. By the argument above,

$$\widehat{x}_j = 0 \text{ for all } j \notin P^s,$$
$$\widehat{x}_j = x_j^{P^s} - \varepsilon_j \text{ for all } j \in J, \text{ and}$$
$$\widehat{x}_j = x_j^{P^s} \text{ for all } j \in P^s \setminus J.$$

Summing up $\widehat{x}_j$ for all $j \in N$ yields $\sum_{j\in N} \widehat{x}_j = \sum_{j\in P^s} x_j^{P^s} - \sum_{j\in J} \varepsilon_j = c - \sum_{j\in J} \varepsilon_j < c$, which contradicts the feasibility of $(\widehat{y}, (\widehat{x}_j)_{j\in N})$. Hence, $(y^{P^s}, (x_i^{P^s})_{i\in N})$ is Pareto efficient.

Secondly, every Nash equilibrium that supports an efficient allocation is a strict Nash equilibrium. Let $s \in S^n$ be a Nash equilibrium that attains an efficient allocation. Denote the set of participants at s by P^s. Since the project is done at efficient allocations, we have $\theta_{P^s} > c$. Furthermore, it is satisfied that $\theta_{P^s} - \theta_i \leq c$ for all $i \in P^s$: if there is an agent $j \in P^s$ such that $\theta_{P^s} - \theta_j > c$, then agent j has an incentive to deviate from s because $\theta_j > \theta_j - x_j^{P^s}$. This contradicts the idea that s is a Nash equilibrium. By Lemma 1 and Remark 1, s is a strict Nash equilibrium. ■

4 Strong equilibria of the participation game

In this section, we characterize the set of strong equilibria and show that there is a strong equilibrium in the participation game. By Lemma 2 and Proposition 1, there is a Nash equilibrium supporting an efficient allocation in the participation game. In the participation game, not all Nash equilibria that support efficient allocations are strong equilibria. If a Nash equilibrium supports an efficient allocation, then the grand coalition does not improve its member payoffs. By the definition of Nash equilibrium, every agent does not have an incentive to deviate from the Nash equilibrium. However, in games with more than two agents, coalitions consisting of more than one and less than n agents may form, and their members may be better off. The following example indicates that the participation game has a Nash equilibrium that supports an efficient allocation, but the Nash equilibrium is not necessarily a strong equilibrium.

Example 2 Let $N = \{1, 2, 3\}$ and let $\theta_1 = \theta_2 = 8$, $\theta_3 = 4$, and $c = 10$. Table 2 shows the payoff matrix of this example. This game has three strict Nash equilibria: $(s_1, s_2, s_3) = (O, I, I)$, (I, O, I), and (I, I, O). All the strict Nash equilibria support efficient allocations. We now focus on the strategy profile $s^* = (I, I, O)$. The payoffs at s^* are $U_1(s^*) = U_2(s^*) = 3$, and $U_3(s^*) = 4$. Suppose that coalition $C = \{2, 3\}$ is formed and deviates from s^*_C to $\widetilde{s}_C = (O, I)$. Note that the public project is undertaken at $(s_1^*, \widetilde{s}_C)$. The payoffs of agents 2 and 3 at $(s_1^*, \widetilde{s}_C)$ are $U_2(s_1^*, \widetilde{s}_C) = 8$ and $U_3(s_1^*, \widetilde{s}_C) = 2/3$, respectively. Hence, the aggregate payoff for C at $(s_1^*, \widetilde{s}_C)$ is $26/3$, which is greater than the sum of payoffs of C at s^*. Therefore, the strategy profile s^* is not a strong equilibrium, while the other strict Nash equilibria are strong equilibria.

⟨ Insert Table 2 here.⟩

In Example 2, the sum of the benefits that participants receive from the project is 12 at every strong equilibrium, which is the smallest sum of the benefits of participants that can be attained in the set of strict Nash equilibria.

4.1 A characterization of strong equilibria

Proposition 2 Let $s^* \in S^n$ denote a strict Nash equilibrium of the participation game, and let P^* be the set of participants at s^*. The strict Nash equilibrium s^* is a strong equilibrium of G if and only if there is no coalition T and its strategy profile $\widehat{s}_T \in S^{\#T}$ such that

$$T_I^* \subsetneq P^*,\ \theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0, \text{and } \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c, \tag{2}$$

where $T_I^* = \{i \in T | s_i^* = I\}$ and $\widehat{T}_I = \{i \in T | \widehat{s}_i = I\}$.

Before proving this proposition, we show the following lemma.

Lemma 3 In the participation game, only the coalitional deviations from a strict Nash equilibrium that satisfy (2) increase the sum of the payoffs to the members of the coalition.

Proof. Let s^* denote a strict Nash equilibrium of the participation game. Denote the set of participants at s^* by P^*. Let T denote a coalition and let $\widehat{s}_T$ denote a profile of strategies for T. Let us denote the set of participants at $(\widehat{s}_T, s_{-T}^*)$ by $\widehat{P}$. If we designate $T_I^* = P^* \cap T$ and $\widehat{T}_I = \widehat{P} \cap T$, then $\widehat{P} = (P^* \setminus (T_I^* \setminus \widehat{T}_I)) \cup (\widehat{T}_I \setminus T_I^*)$. Note that $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*}$.

Claim 1 If $\theta_{\widehat{P}} \geq \theta_{P^*}$, then the deviations by T from s^* are not profitable: $\sum_{i \in T} U_i(s_T^*, s_{-T}^*) \geq \sum_{i \in T} U_i(\widehat{s}_T, s_{-T}^*)$.

Proof of Claim 1. The sum of the payoffs of agents in T at s^* is

$$\theta_T - \frac{\theta_{T_I^*}}{\theta_{P^*}} c > 0, \tag{3}$$

and that at $(\widehat{s}_T, s^*_{-T})$ is

$$\theta_T - \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}} c. \tag{4}$$

Subtracting (4) from (3) yields

$$\begin{aligned}
& -\frac{\theta_{T_I^*}}{\theta_{P^*}} c + \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}} c \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} (\theta_{P^*}\theta_{\widehat{T}_I} - \theta_{\widehat{P}}\theta_{T_I^*}) \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*}\theta_{\widehat{T}_I} - \theta_{T_I^*}\left(\theta_{P^*} - \theta_{T_I^*\setminus\widehat{T}_I} + \theta_{\widehat{T}_I\setminus T_I^*}\right)\right) \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*}\left(\theta_{\widehat{T}_I} - \theta_{T_I^*}\right) - \theta_{T_I^*}\left(\theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}\right)\right).
\end{aligned}$$

Since $\theta_{\widehat{T}_I} - \theta_{T_I^*} = \theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}$, we obtain

$$\frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*} - \theta_{T_I^*}\right)\left(\theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}\right). \tag{5}$$

We have $\theta_{P^*} - \theta_{T_I^*} \geq 0$ because $T_I^* \subseteq P^*$. Since $\theta_{\widehat{P}} \geq \theta_{P^*}$, we obtain $\theta_{\widehat{T}_I\setminus T_I^*} \geq \theta_{T_I^*\setminus\widehat{T}_I}$. Therefore, (5) is greater than or equal to zero. **(End of Proof of Claim 1)**

By Claim 1, the deviations by T satisfy $\theta_{P^*} > \theta_{\widehat{P}}$ if the deviations are profitable. Since $\theta_{P^*} > \theta_{\widehat{P}}$, we obtain $\theta_{T_I^*\setminus\widehat{T}_I} > \theta_{\widehat{T}_I\setminus T_I^*}$.

Claim 2 If $\theta_{\widehat{P}} \leq c$, then the deviations by T are not profitable.

Proof of Claim 2. Note that the project is not undertaken at $(\widehat{s}_T, s^*_{-T})$ if $\theta_{\widehat{P}} \leq c$. Thus, the sum of the payoffs that the members of T receive after the deviation is zero. Since (3) is more than zero, the deviations by T such that $\theta_{\widehat{P}} \leq c$ are not profitable. **(End of Proof of Claim 2)**

From Claim 1 and Claim 2, $\theta_{P^*} > \theta_{\widehat{P}} > c$ must be satisfied so that the deviations by T are profitable. By Lemma 1, $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$. Therefore, $\theta_{P^*} - \theta_{T_I^*\setminus\widehat{T}_I} \leq c$. By Claim 2, $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^*\setminus\widehat{T}_I} + \theta_{\widehat{T}_I\setminus T_I^*} > c$. Thus, we have $\theta_{\widehat{T}_I\setminus T_I^*} > 0$. Accordingly, it follows that $\theta_{P^*} > \theta_{\widehat{P}} > c$ and $\theta_{T_I^*\setminus\widehat{T}_I} > \theta_{\widehat{T}_I\setminus T_I^*} > 0$.

Claim 3 If $T_I^* = P^*$, then the deviations by T are not profitable.

Proof of Claim 3. Note that the difference between the sum of the payoffs that the members of T receive at s^* and that at $(\widehat{s}_T, s^*_{-T})$ is equal to (5). Therefore, the two payoffs are equal if $T_I^* = P^*$. **(End of Proof of Claim 3)**

By Claims 1, 2, and 3, only the deviations by T that satisfy (2) are profitable. ■

Proof of Proposition 2. The sufficiency of the statement is immediate from Lemma 3, and the necessity is trivial. ■

Proposition 2 says that a deviation from a strict Nash equilibrium results in improvements if and only if the following situation exists: at a strict Nash equilibrium, some participants and non-participants form a coalition and can coordinate in a way in which the sum of the benefits from the project of participants decreases and the project is undertaken. In this situation, members of the coalition changing their strategies I to O get benefits, and those who alter O to I suffer losses. However, by transferring part of the benefits to the agents altering O to I, the members switching I to O can make up for the losses. As a result, all members of the coalition can improve their payoffs after this deviation.

From Proposition 2, we confirm that no deviations that increase the sum that participants are willing to pay for the project are profitable. For example, if a coalition deviates from a strict Nash equilibrium in such a way that participants at the strict Nash equilibrium continue to choose participation and some non-participants switch to participation, then the coalitional deviation is not improving. In the participation game with *a public project*, the possibility of monetary transfers decreases the degree of cooperation. This is in contrast with the results of Carraro and Siniscalco (1993), who considered a participation game with *a perfectly divisible public good*. They showed that participants at a strict Nash equilibrium can induce some non-participants at the equilibrium to choose participation by transferring money from the participants to the non-participants if the participants commit themselves to select participation. We conclude from these results that it depends on the type of the public good whether or not the monetary transfers can increase the degree of cooperation.

Shinohara (2004) also analyzed a similar participation game in a mechanism that implements a class of allocation rules including the proportional cost-sharing rule. However, he assumed that monetary transfers among members in coalitions are impossible. He showed that the set of strict Nash equilibria and that of strong equilibria coincide in the participation game without monetary transfers. On the other hand, when monetary transfers are possible, the set of strict Nash equilibria contains that of strong equilibria, and the two sets do not necessarily coincide. Therefore, the set of strong equilibria in the game with monetary transfers is a subset of that in the game without monetary transfers. Proposition 2 points out the possibility that the set of strong equilibria may shrink in the presence of the monetary transfers.

4.2 Existence of a strong equilibrium

Proposition 3 A strong equilibrium exists in the participation game with a public project.

Proof. Let P^{min} be such that $P^{min} \in \arg\min_{P\subseteq N} \theta_P$ subject to $\theta_P > c$. Since P^{min} satisfies $\theta_{P^{min}} - \theta_i \leq c$ for every $i \in P^{min}$, P^{min} is supportable as a strict Nash equilibrium. Let $s^{min} \in S^n$ be the strict Nash equilibrium at which P^{min} is the set of participants. We show that s^{min} is a strong equilibrium. By Proposition 2, it is sufficient to show that there is no deviation that satisfies (2).

Let T be a coalition and let s_T be a profile of strategies for T. Let us denote $T_I^{min} = \{i \in T | s_i^{min} = I\}$ and $T_I = \{i \in T | s_i = I\}$. Note that the set of participants at (s_T, s_{-T}^{min}) is $(P^{min} \cup (T_I \backslash T_I^{min})) \backslash (T_I^{min} \backslash T_I)$. Let us define $\widetilde{P} := (P^{min} \cup (T_I \backslash T_I^{min})) \backslash (T_I^{min} \backslash T_I)$.

If T deviates in a way that satisfies $\theta_{\widetilde{P}} > c$, then we must have $\theta_{\widetilde{P}} \geq \theta_{P^{min}} > c$ because $\theta_{P^{min}}$ is the smallest sum of participants' benefits that is attained at strict Nash equilibria. Then, we have $\theta_{T_I^{min} \backslash T_I} \leq \theta_{T_I \backslash T_I^{min}}$, which indicates that T can not deviate in a way that satisfies (2). Theorefore, s^{min} is a strong equilibrium of the participation game. ■

From Proposition 2, the set of strict Nash equilibria contains that of strong equilibria, but the converse is not always true. However, in the case of identical agents, every strict Nash equilibrium is a strong equilibrium.

Corollary 1 Suppose that agents are identical: $\theta_i = \theta_j$ for all pairs of agents $\{i, j\}$. Then, all strict Nash equilibria are strong equilibria in the participation game.

Proof. Let $\theta = \theta_i$ for all $i \in N$ and let P be a set of participants that is supported as a strict Nash equilibrium. By Lemma 1, P satisfies $\#P\,\theta > c$ and $(\#P - 1)\theta \leq c$, or $\dfrac{c}{\theta} < \#P \leq \dfrac{c}{\theta} + 1$. Since $\#P$ is a natural number, we find from these inequalities that $\#P$ is unique. Therefore, $\#P\,\theta$ is the smallest sum of the benefits that participants receive from the project in the set of strict Nash equilibria. In the proof of Proposition 3, we show that a strict Nash equilibrium at which the sum of the benefits of the participants is the smallest in the set of strict Nash equilibria is strong. Thus, P is attained at a strong equilibrium of the game. ■

Although the set of Nash equilibria and that of strong equilibria are subsets of that of Nash equilibria, it is not evident whether the two sets have an inclusion relation. In the participation game, strict Nash and strong equilibria are both non-empty, and the set of strong equilibria is included in that of strict Nash equilibria. This is an interesting aspect of our model.

The results of this paper contrast with those of a participation game with a perfectly divisible public good. In the participation game with a perfectly divisible public good, Nash equilibria frequently support inefficient allocations, and there is not necessarily a strong equilibrium (Saijo and Yamato, 1999; Shinohara, 2003). In addition, in the standard game of the voluntary contribution to a perfectly divisible public good, Nash-equilibrium allocations are not efficient. Hence, a strong equilibrium does not exist in the voluntary contribution game. However, in the participation game with a public project, there is a Nash equilibrium that supports an efficient allocation. Moreover, there is a strong equilibrium in the participation game, and only efficient allocations can be attained at strong equilibria. This is another interesting aspect of our model.

The following theorem summarizes the results that have been obtained so far.

Theorem In the participation game with a public project, (i) there is a Nash equilibrium at which the efficiency of an allocation is achieved, (ii) the set of Nash equilibria that supports efficient allocations coincides with the set of strict Nash equilibria, (iii) a strong equilibrium exists, and (iv) the set of strict Nash equilibria includes that of strong equilibria, but the converse inclusion relation does not necessarily hold.

Remark 2 Let us consider the participation game in which the project is undertaken if and only if the sum of the benefits that participants receive from the project is more than or equal to the cost c: for all sets of participants P, $\theta_P \geq c$ if and only if $y^P = 1$. In this participation game, there are not necessarily strict Nash equilibria. However, the game has a Nash equilibrium at which efficient allocations are attained. Every set of participants at Nash equilibria that support efficient allocations is characterized as $P \subseteq N$ with $\theta_P \geq c$ and $\theta_P - \theta_i < c$ for all $i \in P$. We can show that the game has a strong equilibrium and the set of strong equilibria is included in that of Nash equilibria that supports efficient allocations in this participation game.

5 Participation games with a multi-unit public good

5.1 A participation game in which at most two units of the public good can be produced

In this section, we consider a participation game with a multi-unit public good. The participation game with a multi-unit public good consists of two stages. In the first stage, agents simultaneously choose I or O, and the agents that select I choose the level of the public good and share the cost of the public good in the second stage. However, in the participation game with a multi-unit public good, the level of the public good is assumed to take zero, one , or two. Let Y be a *public good space* such that $Y = \{(y_1, y_2) \in \{0,1\}^2 | y_1 \geq y_2\}$: if $y_1 = y_2 = 1$, then two units of the public good are produced; if $y_1 = 1$ and $y_2 = 0$, then one unit of the public good is produced; if $y_1 = y_2 = 0$, then zero units of the public good are produced. Let $c > 0$ be a constant cost of producing one unit of the public good. Let $\theta_i^k > 0$ denote agent i's marginal benefit from the k-th unit of the public good. Each agent i has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$. We denote $\theta_P^k = \sum_{j \in P} \theta_j^k$ for all $k \in \{1, 2\}$ and for all $P \subseteq N$. Let us assume that $\theta_i^1 > \theta_i^2$ for all $i \in N$ and $\theta_N^2 > c$. This implies that two units of the public good are produced at every Pareto efficient allocation.

The following is the assumption regarding to the second-stage outcomes.

Assumption 4 Let P be a set of participants, and let $(y^P, (x_j^P)_{j \in P})$ be the allocation for the participants. Let us assume that the allocation satisfies the following conditions.

(C.1) $y^P = \max\{k \in \{0, 1, 2\} \,|\, \theta_P^k - c > 0\}$. (Surplus Maximization)
(C.2) $\sum_{j \in P} x_j^P = y^P c$. (Budget Balance)
(C.3) $V_i(y^P, x_i) \geq 0$ for every $i \in P$. (Individual Rationality)
(C.4) $x_i^P > 0$ for every $i \in P$. (Positive Cost Burden)

From (C.1), the participants produce the public good in a way that maximizes the surplus of the participants. The cost of the public good is distributed in a way that satisfies the conditions of budget balance and individual rationality. Every participant shares a positive cost.

Many allocation rules satisfy (C.1), (C.2), (C.3), and (C.4). For example, the unit-by-unit proportional cost-sharing rule introduced by Yu (2005) satisfies all the conditions.*[1] Yu (2005) constructed the mechanism that implements the unit-by-unit cost-sharing rule.

In the participation game with a multi-unit public good, there is not necessarily a Nash equilibrium at which efficient allocations are attained. In the following example, there is no Nash equilibrium that supports efficient allocations, and strong equilibria do not exist.

Example 3 Let $N = \{1, 2, 3, 4\}$. Suppose that $\theta_i^1 = 2$ and $\theta_i^2 = 0.8$ for all $i \in N$ and $c = 1$. The cost of the public good is assumed to be distributed equally among participants. Let P be a set of participants. Note that one unit of the public good is produced if $\#P = 1$, and two units of the public good are provided if $\#P \geq 2$. Table 3 shows the payoffs to participants and non-participants in this example. From the table, we can easily find that one and only one agent selects participation at every strict Nash equilibrium. However, these Nash equilibria are not strong equilibria, since three non-participants at the Nash equilibrium can gain higher payoffs if all of them jointly deviate from non-participation to participation; thus, a strong equilibrium does not exist in this example.

⟨Insert Table 3 here.⟩

5.2 Existence of Nash equilibria that support efficient allocations

In this subsection, we investigate whether or not a Nash equilibrium supports an efficient allocation in the participation game with a multi-unit participation game. For this, we first characterize the set of Nash equilibria at which two units of the public good are produced.

Proposition 4 Let P be a set of participants. The set of participants P is supported as a Nash equilibrium and two units of the public good are provided at the equilibrium if and only if $P \subseteq N$ satisfies (i) $\theta_P^2 > c$, (ii) $\theta_P^2 - \theta_i^2 \leq c$ for all $i \in P$, and (iii) if there is an agent $i \in P$ such $\theta_P^1 - \theta_i^1 > c$, then $\theta_i^2 \geq x_i^P$.

*[1] For every unit of the public good, the unit-by-unit proportional cost-sharing rule allocates the cost proportional to each agent's willingness to pay for that unit.

Proof. **(sufficiency)** Let P denote a set of participants that satisfies (i), (ii), and (iii). By (i), two units of the public good are produced if P is a set of participant. By conditions (ii) and (iii), no agent $i \in P$ have an incentive to switch I to O. Clearly, no agents $i \notin P$ have an incentive to choose I, given the participation of P. Hence, P is a set of participants that is supportable as a Nash equilibrium.
(necessity) Suppose that a set of participants P is supported as a Nash equilibrium and two units of the public good are provided at the equilibrium. Since two units of the public good are provided, condition (i) must be satisfied. If P do not satisfy (ii), then there exists agent i such that $\theta_P^2 - \theta_i^2 > c$. Hence, agent i has an incentive to deviate from I to O, which is a contradiction. Suppose that there is an agent $i \in P$ such that $\theta_P^1 - \theta_i^1 > c$ and $\theta_i^2 < x_i^P$. Then, he obtains the payoff θ_i^1 if he chooses O, and he receives the payoff $\sum_{k=1}^{2} \theta_i^k - x_i^P$ if he chooses I. Since $\theta_i^2 < x_i^P$, he has an incentive to switch from I to O. This is a contradiction. Therefore, P satisfies (i), (ii), and (iii). ■

We determine whether or not two units of the public good are produced at a Nash equilibrium. First, consider the following case:

Case 1 There exists a set of participants P such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$.

Proposition 5 In Case 1, there is a set of participants that is supported as a Nash equilibrium of the participation game.

Proof. Let $P \subseteq N$ be such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. Note that $\theta_P^2 - \theta_i^2 \leq \theta_P^1 - \theta_i^1 \leq c$ for every $i \in P$, in which the first inequality holds with equality if $P \setminus \{i\}$ is empty. Hence, P is a Nash-equilibrium set of participants, and two units of he public good are provided.

■

Next, we proceed with our analysis in Case 2:

Case 2 For every $P \subseteq N$, if P satisfies $\theta_P^2 > c$, then $\theta_{P\setminus\{i\}}^1 > c$ for some $i \in P$.

Proposition 6 Let $P \subseteq N$ be a set of participants such that $\theta_{P\setminus\{i\}}^1 > c$ for *every* $i \in P$ and $\theta_P^2 > c$. Then, P is a Nash-equilibrium set of participants if and only if $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for every $i,\ j \in P$.

Proof. Let $P \subseteq N$ be a set of participants such that $\theta^1_{P\setminus\{i\}} > c$ for every $i \in P$ and $\theta^2_P > c$. Since $\theta^1_{P\setminus\{i\}} > c$ for every $i \in P$, we have $\#P \geq 2$.

(sufficincy) Let us suppose that P satisfies $\#P = 2$, $\theta^2_i = \theta^2_j = c$, and $x^P_i = x^P_j = c$ for every $i,\ j \in P$. Then, we have $V_i(y^P, x^P_i) = \theta^1_i + \theta^2_i - x^P_i = \theta^1_i$ and $V_i(y^{P\setminus\{i\}}, x^{P\setminus\{i\}}_i) = \theta^1_i$ for every $i \in P$. From these conditions, P is supported as a Nash equilibrium of the participation game.

(necessity) Suppose that P is supported as a Nash equilibrium of the participation game. Suppose, without loss of generality, that $P = \{1, 2, \ldots, l\}$, in which $l \geq 2$, and $\theta^2_1 \geq \theta^2_2 \geq \cdots \geq \theta^2_l$. Then, there is $\alpha_i \in (0, 1]$ for every $i \in P$ such that $\theta^2_i = \alpha_i \theta^2_1$. Note that $1 = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l > 0$.

From Proposition 4, P satisfies the following conditions:

$$\sum_{i\in P} \alpha_i \theta^2_1 > c. \tag{6}$$

$$\left(\sum_{i\in P} \alpha_i - \alpha_j\right) \theta^2_1 \leq c \text{ for every } j \in P. \tag{7}$$

$$\alpha_j \theta^2_1 \geq x^P_j \text{ for every } j \in P. \tag{8}$$

We obtain from (6) that $\theta^2_1 > c/\sum_{i\in P} \alpha_i$. Since $\sum_{i\in P} \alpha_i - \alpha_l \geq \sum_{i\in P} \alpha_i - \alpha_j$ for every $j \in P$, (7) implies $\theta^2_1 \leq c/(\sum_{i\in P} \alpha_i - \alpha_l)$. It follows from (8) that $\theta^2_1 \geq x^P_j/\alpha_j$ for every $j \in P$. By these conditions, we must have

$$\frac{c}{\sum_{i\in P\setminus\{l\}} \alpha_i} - \frac{x^P_j}{\alpha_j} \geq 0 \text{ for every } j \in P \tag{9}$$

so that P satisfies (6), (7), and (8). It follows from (9) that

$$\frac{1}{\alpha_j \sum_{i\in P\setminus\{l\}} \alpha_i} \left(\alpha_j c - x^P_j \sum_{i\in P\setminus\{l\}} \alpha_i\right) \geq 0 \text{ for every } j \in P.$$

We obtain from these conditions that $\alpha_j c - x^P_j \sum_{i\in P\setminus\{l\}} \alpha_i \geq 0$ for every $j \in P$. Summing up these conditions for every $j \in P$ yields

$$\sum_{j\in P} \alpha_j c \geq \left(\sum_{j\in P} x^P_j\right) \left(\sum_{i\in P\setminus\{l\}} \alpha_i\right) = 2c \sum_{i\in P\setminus\{l\}} \alpha_i.$$

Therefore, we have

$$\alpha_l \geq \sum_{i \in P \setminus \{l\}} \alpha_i. \tag{10}$$

First, we prove that $\#P = 2$ and $\theta_i^2 = \theta_j^2$ for all $i,\ j \in P$. Suppose, on the contrary, that $\#P \geq 3$. Since $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$, we have $\alpha_l < \sum_{j \in P \setminus \{l\}} \alpha_j$. This is a contradiction. Therefore, it follows that $\#P = 2$, which indicates $l = 2$ and $\alpha_2 \geq \alpha_1$. Condition $\alpha_2 \geq \alpha_1$, together with $\alpha_1 \geq \alpha_2$, implies $\alpha_1 = \alpha_2$. Therefore, we have $\theta_1^2 = \theta_2^2$.

Secondly, we show that $\theta_1^2 = \theta_2^2 = c$ and $x_1^P = x_2^P = c$. Let us define $\theta^2 = \theta_1^2 = \theta_2^2 = c$. Since P is a Nash-equilibrium set of participants, θ^2 satisfies $2\theta^2 > c$, $\theta^2 \leq c$, $\theta^2 \geq x_1^P$, and $\theta^2 \geq x_2^P$. From the first two conditions, we have $\theta^2 \in (c/2, c]$. It must be satisfied that $x_1^P \leq c$ and $x_2^P \leq c$, because θ^2 takes at most c. Since $x_1^P + x_2^P = 2c$, we have $x_1^P = x_2^P = c$. Then, $\theta^2 = c$ must be satisfied in order that P is supported as a Nash equilibrium of the participation game. ■

Corollary 2 Suppose that agents' preferences are identical: for every $k \in \{1, 2\}$ and for every $i,\ j \in N$, $\theta_i^k = \theta_j^k$. Then, in Case 2, there is a Nash equilibrium that supports efficient allocations if and only if there is a set of participants P such that $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for all $i,\ j \in P$.

Proof. Suppose that agents' preferences are identical. Then, for every $P \subseteq N$ with $\theta_P^2 > c$, if $\theta_P^1 - \theta_i^P > c$ for some $i \in P$, then $\theta_P^1 - \theta_i^P > c$ for every $i \in P$. Thus, it follows from Proposition 6 that there is a Nash equilibrium that supports efficient allocations if and only if there is a set of participants P such that $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for all $i,\ j \in P$. ■

From Proposition 5 and Corollary 2, we can demonstrate that, in the case of identical agents, the public good is less likely provided efficiently if the participation of many agents is needed for the efficient provision of the public good. Suppose that all agents receive a (marginal) benefit $\theta^k > 0$ from the k-th unit of the public good for every $k \in \{1, 2\}$. Let $(\theta^1, \theta^2) \in \mathbb{R}_{++}^2$ be a profile of the marginal benefits. Note that $\theta^1 > \theta^2$. If a Nash equilibrium exists such that $p \geq 1$ agents enter the mechanism and two units of the public good are produced, then we have $p\theta^2 > c \geq (p-1)\theta^2$. Note that $p\theta^1 > p\theta^2 > c$, but it is not clear whether $(p-1)\theta^1 > c$ holds or not. From Proposition 5, the participation of p agents is attained at a Nash equilibrium if $(p-1)\theta^1 \leq c$. Hence, if (θ^1, θ^2) satisfies $p\theta^2 > c \geq (p-1)\theta^1$, then a Nash equilibrium supports the participation of p agents. If $(p-1)\theta^1 > c$, then the participation of p

agents is attained at a Nash equilibrium if and only if $p = 2$, $\theta^2 = c$, and all of p agents pay c from Corollary 2. From these conditions, we can show the set of parameters at which the public good is provided efficiently in Figures 1, 2, and 3.

In the case of $p = 1$, one agent chooses participation, and two units of the public good are provided in a Nash equilibrium if and only if $\theta_2 > c$. The set of profiles (θ^1, θ^2) that satisfies $\theta_2 > c$ are shown in Figure 1. In this case, the public good is produced efficiently if and only if (θ^1, θ^2) is in the shaded area of Figure 1. In the case of $p = 2$, the efficient provision of the public good is achieved if either one of the following conditions is satisfied:

(i) $\theta^1 \leq c$ and $\theta^2 > \dfrac{c}{2}$

(ii) $\theta^1 > \theta^2 = c$

The set of (θ^1, θ^2) that satisfies (i) or (ii) is depicted in Figure 2: (θ^1, θ^2) in the shaded triangle satisfies (i), and B in Figure 2 is the set of preference parameters that satisfy (ii). In the case of $p \geq 3$, the efficient provision of the public good is supportable as a Nash equilibrium if and only if (θ^1, θ^2) satisfies $\theta^1 \leq \dfrac{c}{p-1}$ and $\theta^2 > \dfrac{c}{p}$. The shaded area in Figure 3 is the range of (θ^1, θ^2) at which the public good is provided efficiently.

Note that, under the condition of $(p-1)\theta^1 > c$, the efficient provision of the public good is attained at an equilibrium only if $p = 2$. From Corollary 2, a Nash equilibrium supports the efficient provision of the public good only if $\theta^2 = c$, even though $p = 2$ and $(p-1)\theta^1 > c$ are satisfied. Thus, the range of the efficient provision of the public good consists largely of (θ^1, θ^2), which satisfies $p\theta^2 > c \geq (p-1)\theta^1$ and is depicted as a triangular area in Figures 2 and 3. Note that (θ_1, θ_2) in these triangular areas satisfies

$$1 > \frac{\theta^2}{\theta^1} > \left(\frac{c}{p}\right) \bigg/ \left(\frac{c}{p-1}\right) = 1 - \frac{1}{p}.$$

Since $1 - \dfrac{1}{p}$ converges to 1 as the number of participants p becomes large, the range of the diminishing rate of marginal benefits $\dfrac{\theta^2}{\theta^1}$ shrinks as the number of participants increases. Thus, when the set of participants consists of many agents, the diminishing rate of the marginal benefits must be low for the set to be supported as a Nash equilibrium. We can say from this result that the public good is less likely to be efficiently provided if the participation of many agents is needed for the efficient provision of the public good.

⟨ Insert Figures 1, 2, and 3 here. ⟩

In this section, we confirm that there is not necessarily a Nash equilibrium to support efficient allocations in the participation game with a multi-unit public good. Moreover, it is difficult to achieve allocative efficiency when the participation of many agents is needed to produce the public good efficiently. These results indicate that strategic behavior in participation decisions often leads to inefficient allocations, even though a mechanism is constructed in such a way as to implement an efficient allocation rule. An implication that is similar to Saijo and Yamato (1999) can be derived even in the participation game in which up to two units of the public good can be produced.

Remark 3 In Proposition 6, we characterized a Nash-equilibrium set of participants P under the condition that $\theta_P^1 - \theta_i^1 > c$ for every $i \in P$. This condition holds only if agents' preferences are identical or slightly different. However, a set of participants does not necessarily satisfy this condition if the set of participants is composed of agents who have different preferences. Thus, in the case of heterogeneous agents, there may be a set of participants P that satisfies $\theta_P^2 > c$, $\theta_{P\setminus\{i\}}^2 \leq c$ for every $i \in P$, $\theta_{P\setminus\{i\}}^1 > c$ for some $i \in P$, and $\theta_{P\setminus\{j\}}^1 < c$ for some $j \in P$. The following examples indicate that such sets of participants may or may not be Nash-equilibrium sets of participants in the participation game, depending on the preference parameters of the participants.

Example 4 Let $N = \{1, 2\}$ and let $\theta_1^1 = 40$, $\theta_1^2 = 9$, $\theta_2^1 = 7$, $\theta_2^2 = 6$, and $c = 10$. In this example, the costs of producing the public good are distributed according to a unit-by-unit public good among participants: for every unit of the public good, the unit-by-unit proportional cost-sharing rule allocates the cost proportional to each agent's willingness to pay for that unit. Two units of the public good are produced only if two agents choose I, and one unit of the public good is provided only when agent 1 chooses I and agent 2 chooses O. If agent 1 and agent 2 choose I, then the payoff of agent 1 is $\theta_1^1 + \theta_1^2 - \dfrac{\theta_1^1}{\theta_1^1 + \theta_2^1} c - \dfrac{\theta_1^2}{\theta_1^2 + \theta_2^2} c = \dfrac{1621}{47} \approx 34.49$, and that of agent 2 is $\theta_2^1 + \theta_2^2 - \dfrac{\theta_2^1}{\theta_1^1 + \theta_2^1} c - \dfrac{\theta_2^2}{\theta_1^2 + \theta_2^2} c = \dfrac{353}{47} \approx 7.51$. Table 4 is the payoff matrix of this example. In this example, there is a Nash equilibrium at which two agents choose I and two units of the public good are provided.

⟨ Insert Table 4 here.⟩

Example 5 Consider a two-agent participation game in which $\theta_1^1 = 12$, $\theta_1^2 = 8$, $\theta_2^1 = 8$, $\theta_2^2 = 6$, and $c = 10$. As in Example 4, the costs of producing the public good are allocated according to the unit-by-unit public good among participants. The payoff matrix is shown in Table 5. In this game, there is only one Nash equilibrium at which agent 1 chooses I and agent 2 chooses O. At the equilibrium, one unit of the public good is produced, and an inefficient allocation arises.

⟨ Insert Table 5 here.⟩

Agents have greatly different preferences for the public good in Example 4, while they have relatively similar preferences in Example 5. From these examples, it seems valid to conjecture that a set of participants is attained at a Nash equilibrium if the agents' preferences differ greatly.

6 Conclusion

We have investigated a participation game in the provision of a discrete public good. First, we examined a case of a public project. We showed that there are Nash equilibria that achieve allocative efficiency, and some efficient allocations are attained at strong equilibria in the participation game with a public project. Secondly, we examined a case in which at most two units of the public good are provided. In this case, there is not necessarily a Nash equilibrium that supports an efficient allocation. We proved that, in the case of identical agents, the set of participants consisting of many agents is less likely a Nash-equilibrium set of participants. Therefore, the efficient provision of the public good is rarely achieved if the participation of many agents is needed for the efficient provision of the public good in the case of identical agents. We found from these results that the assumption that only one unit of the public good can be produced plays a significant role in the existence of a Nash equilibrium that supports efficient allocations and that of a strong equilibrium. We also concluded that strategic behavior in participation decisions leads to inefficiency of the allocations even in a participation game in which at most two units of the public good can be produced.

In the case of heterogeneous agents, it is unclear which sets of participants are supported as Nash equilibria and which conditions guarantee the efficient provision of the public good at Nash equilibria. Future studies will be needed to establish conditions under which efficient allocations are attained at Nash equilibria.

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	I	O
I	$5/12,\ 5/12,\ 5/12$	$1/4,\ 3/4,\ 1/4$
O	$3/4,\ 1/4,\ 1/4$	$0,\ 0,\ 0$

I

	I	O
I	$1/4,\ 1/4, 3/4$	$0,\ 0,\ 0$
O	$0,\ 0,\ 0$	$0,\ 0,\ 0$

O

Table. 1 Payoff matrix of Example 1

	I	O
I	4, 4, 2	4/3, 8, 2/3
O	8, 4/3, 2/3	0, 0, 0

I

	I	O
I	3, 3, 4	0, 0, 0
O	0, 0, 0	0, 0, 0

O

Table. 2 Payoff matrix of Example 2

The number of participants	Payoffs to participants	Payoffs to non-participants
0	-	0
1	1	2
2	1.8	2.8
3	32/15	2.8
4	2.3	-

Table. 3 Payoffs of Example 3

1 \ 2	I	O
I	34.49, 7.51	30, 7
O	0, 0	0, 0

Table. 4 The payoff matrix of Example 4

2 \ 1	I	O
I	8.29, 5.71	2, 8
O	0, 0	0, 0

Table. 5 The payoff matrix of Example 5

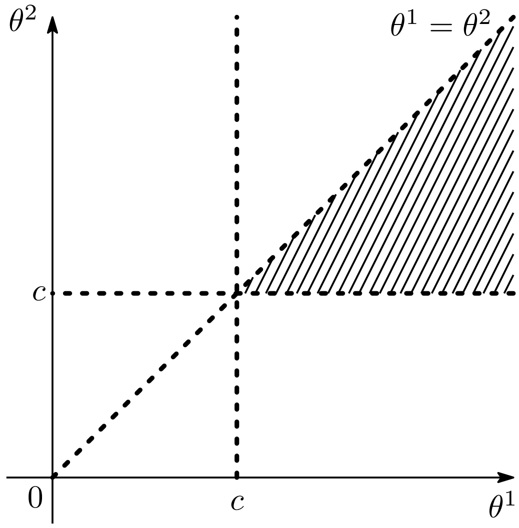

Fig. 1 In the case of $p = 1$, the efficient provision of the public good is achieved in the shaded area $\left\{(\theta^1, \theta^2) \in \mathbb{R}^2_{++} | \theta^1 > \theta^2 > c\right\}$.

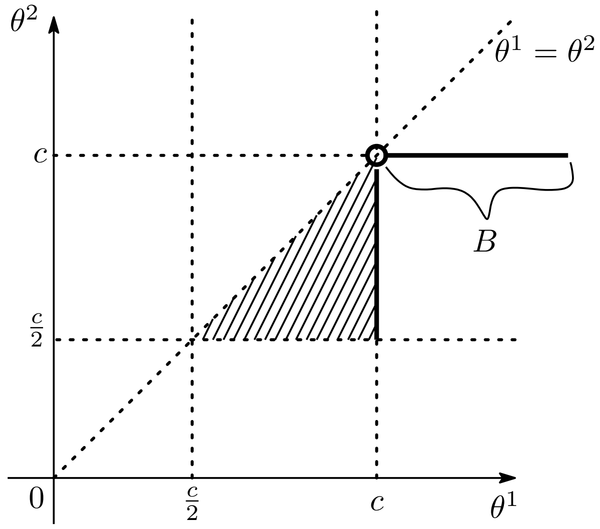

Fig. 2 In the case of $p = 2$, the efficient provision of the public good is achieved in the shaded triangle $\left\{(\theta^1, \theta^2) \in \mathbb{R}^2_{++} | c \geq \theta^1 > \theta^2 > \dfrac{c}{2}\right\}$ or $B = \left\{(\theta^1, \theta^2) \in \mathbb{R}^2_{++} | \theta^1 > \theta^2 = c\right\}$.

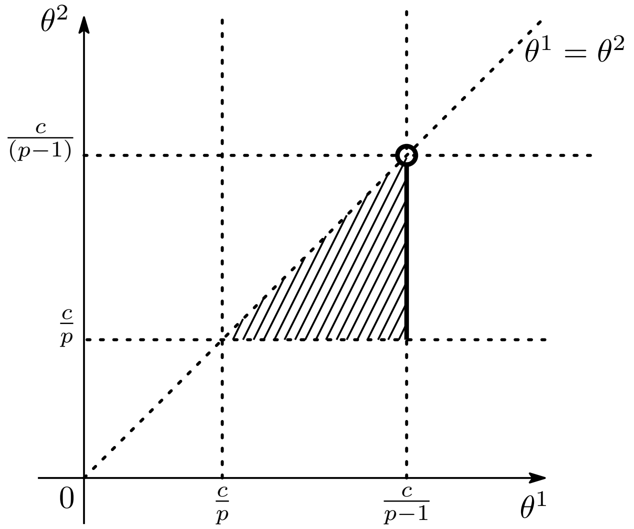

Fig. 3 In the case of $p \geq 3$, the efficient provision of the public good is achieved in $\left\{(\theta^1, \theta^2) \in \mathbb{R}^2_{++} | \dfrac{c}{p-1} \geq \theta^1 > \theta^2 > \dfrac{c}{p}\right\}$.